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Low-frequency phenomena in dynamical systems with many attractors

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Power spectra of multistable systems have a low-frequency component corresponding to jumps among independent attractors. This situation is modeled with a one-dimensional cubic map disturbed by noise. Besides the dependence of the spectral slope on noise, we show that the Lyapunov exponent is made of two separate contributions corresponding to the attracting and repulsive regions, respectively, each weighted with its relative occupation time.

Recent experiments[1,2] on nonlinear driven systems yield power spectra with a low-frequency divergence $f^{-\alpha}$ with α around 1, whenever the following conditions are fulfilled: (i) the system has at least two independent attractors and (ii) it is "open" to external fluctuations which cause jumps among the attractors which have different degrees of stability.

Previous models of low-frequency features (besides the high-frequency spectra associated with chaos) are of two types. The first one is a deterministic diffusion between different subregions of a single attractor.[3,4] The second one is represented by switches between two states which gives a single Lorentzian spectrum.[5] Both models are insufficient to explain the above experiments since they yield fixed slopes [as, e.g., $f^{-1/2}$ in Ref. 3(a) or f^{-2} in Ref. 5], while experiments[1,2] show variations in the slope depending on the strength of the applied noise.

In this paper we present the theory of jumps over independent attractors, triggered by external noise. At variance with Refs. 3 and 4, noise plays an essential role. At variance with Ref. 6, the dynamics involves more than two objects (in this case three: two attractors and a repulsive region).

Furthermore, we find that noise is not essential for those attractors which have just become unstable having undergone a crisis.[6] In such a latter case, previously separate basins of attraction have merged yielding deterministic diffusion between subregions of a single attractor. Details will be reported separately.

A dynamics in terms of a recursive map must allow for at least two independent attractors.[7] The simplest one-dimensional map with two attractors must have two extrema.[8] Hence we study a cubic map in the interval $(-1,1)$,

$$X_{n+1}=(a-1)X_n-aX_n^3 \ . \tag{1}$$

To account for item (iii) the map will be disturbed by additive white noise with rms between 10^{-7} and 10^{-5}. Up to a value $a=\bar{a}=3\sqrt{3}/2+1=3.598\,076\ldots$, the motion is confined either on the interval $(-1,0)$ or $(0,1)$ with qualitative features like the well-known logistic map. For $a=\bar{a}$, we may still have two independent attractors whose domains, however, are interlaced in complicated ways over the interval $(-1,1)$. For $a=4$, even this new structure undergoes a crisis and there are no longer attractors.

The simplest stable pair of attractors A,B above $\bar{a}$ is a pair of period-3 attractors which are superstable for $a_s=3.981\,797\,394\ldots.$ These period-3 attractors undergo a crisis for $a=\tilde{a}=3.982\,000\,642\ldots.$ For $a_s<a<\tilde{a}$ [Fig. 1(a)], the presence of a small amount of noise makes it easy to leave one attractor and jump toward the other one. Before landing onto the other attractor, the representative point wanders on the available space through a long transient, because of the complex structure of the two basins of attraction. Such a situation is shown in Fig. 2(a) and the corresponding low-frequency power spectra, for different noise levels, are given in Fig. 2(b). These spectra show a

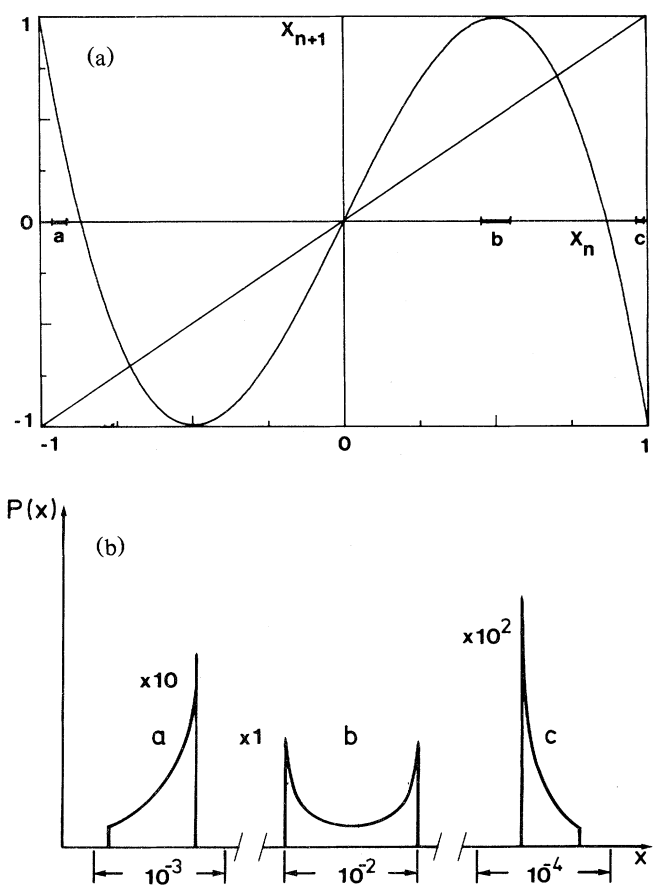

FIG. 1. Cubic map for $\tilde{a}$. *a*, *b*, and *c* show (not in scale) the intervals covered by one of the two period-3 attractors. The lower part shows the asymptotic density on the three intervals.

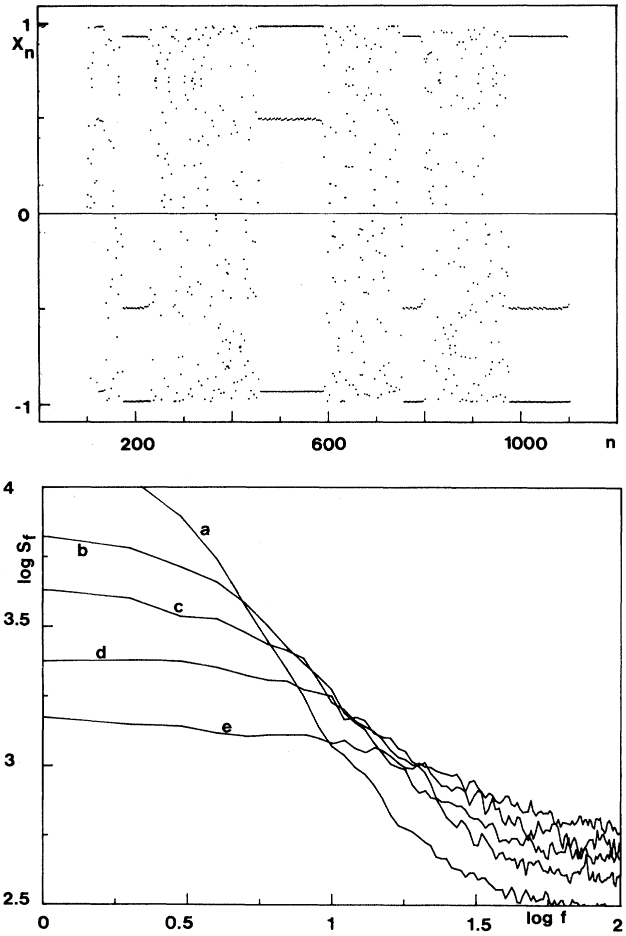

FIG. 2. Time and frequency data for $\tilde{a}$. The time sequence (taken for $\sigma = 10^{-6}$) shows alternation between attractive and repulsive regions. The power spectra ($\langle |x(f)|^2 \rangle$) are given for increasing noise levels σ, that is; curve a, 5×10^{-7}; curve b, 10^{-6}; curve c, 2×10^{-6}; curve d, 4×10^{-6}; curve e, 10^{-5}; they can be fitted by f , with α decreasing from 1.5 for curve a to 0.5 for curve e. Notice that the order of the curves (a–e) is inverted in the high-log$f$ part.

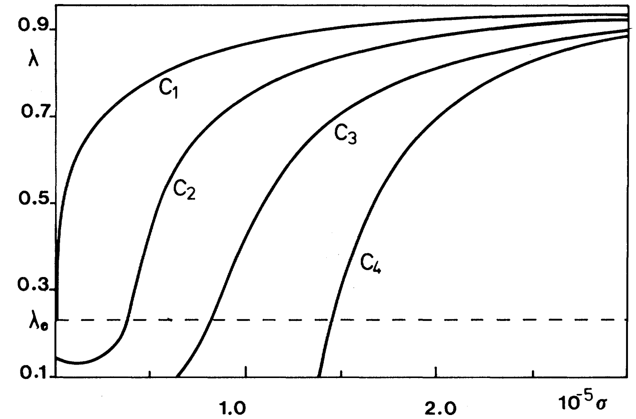

FIG. 3. Lyapunov exponents λ vs noise σ for different a values. C_1: $a=\tilde{a}$, C_2: $a=3.98197$; C_3: $a=3.98193$; C_4: $a=a_0$.

power-law region extending over about one decade with a slope between 0 and 2. They appear qualitatively in agreement with the experimental spectra of Refs. 1 and 2.

Figure 3 shows the experimental dependence of the Lyapunov exponent on the rms σ of the applied noise for different values of the control parameter a. The relevant λ range is confined between the value $\lambda_0 = \ln 2/3$ where the two period-3 attractors undergo a crisis and $\lambda_T = \ln 3$, where the whole map undergoes a crisis. The added noise allows for escape from an attracting region whenever the phase point is away from the boundary by less than about 3σ. For any a value, the corresponding $\sigma = \sigma_0(a)$ for which the λ curve crosses $\lambda = \lambda_0$ is approximately one-third the minimum distance between the border of the region covered by the attractor and the frontier of the immediate attraction domain.[9] Experimentally the dependence $\sigma_0(a)$ is fitted by

$$\sigma_0(a) = \sigma_M \left[1 - \left(\frac{a - a_0}{\tilde{a} - a_0} \right)^{\gamma} \right] , \qquad (2)$$

where $\sigma_M = 1.45\times10^{-5}$, $a_0 = 3.98184$, $\gamma = 1.40 \pm 0.1$, and $a_0 < a < \tilde{a}$. In other words, the jumps between attractors appear as a noise-induced crisis. σ_T is the noise value for which the whole map undergoes a crisis; hence all the λ curves cluster at λ_T for $\sigma = \sigma_T$ with an empirical scaling law $\bar{\lambda} = F(a, \sigma)$, where $\bar{\lambda} = (\lambda - \lambda_0)/(\lambda_T - \lambda_0)$. During the motion, when the phase point is on an attractor, the Lyapunov exponent is practically equal to λ_0, and when it is in the transient region, we can appropriately take the asymptotic crisis value λ_T. We can then split the sum defining λ into separate contributions. Calling N_A and N_t the number of steps on the attractor and on the transient, respectively, we get

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \left[2 \sum_{k=1}^{N_A} \ln |f'(x_K)| + \sum_{k=1}^{N_t} \ln |f'(x_K)| \right]$$

$$= 2k\lambda_0 + k'\lambda_T , \qquad (3)$$

where $k = \lim_{N\to\infty} N_A/N$ and $k' = \lim_{N\to\infty} N_t/N$, with $2k + k' = 1$, are the fractional times spent on the attractor and on the transient, respectively. Notice from Eq. (3) that the fractional time k' spent on the transient coincides with the reduced Lyapunov exponent $\bar{\lambda}$ given in the scaling law. Inspection of the data of Figs. 2(b) and 3 shows that the slopes α of the $f^{-\alpha}$ regions scale linearly with $\bar{\lambda}$ as

$$\alpha = 2(1 - \bar{\lambda}) \qquad (4)$$

from $\alpha = 2$ (single Lorentzian), when $\bar{\lambda} = 0$ (no transient), to $\alpha \to 0$ (flat spectrum) for $\bar{\lambda} = 1$ (all transient).

A model explanation of these spectra can be built by a simplifying assumption. The system coordinate x_n at time $n = t$ (for simplicity we shall write the time as a continuous variable) is either $V_{\pm1}(t)$ or $V_0(t)$, where $V_{\pm1}$ are the values assumed on the first or on the second attractor, and V_0 that on the transient region. The probability of escape from an attractor, under the influence of noise, depends on its distance from the border of the immediate basin of attraction.[9] Figure 1(b) shows the density distribution within the attractor for $a = \tilde{a}$: noise amplitudes around $\sigma_0(a)$ destroy any deterministic structure, making that picture qualitatively valid also for $a < \tilde{a}$. The region of escape is mainly the smallest one; however, since the point is on that segment every three iterations, we can take a uniform escape rate for time longer than such a period. For the reentrance, since points in the transient region are rapidly scrambled over the whole interval $(-1,1)$, the reentrance probability is maximum for the largest attracting interval and it is practically independent from the actual position V_0 within the transient region.

With the above assumptions, the probabilities $p_{\pm 1}(t)$ of being in the two attractors have a constant death rate a and a birth rate b. Taking $T=bt$, $a/b=\eta$, they obey the simple rate equations (the dot stays for d/dT)

$$\dot{p}_{\pm 1}=-\eta p_{\pm 1}+p_0 \ , \tag{5}$$

with the normalization $p_{+1}+p_{-1}+p_0=1$.

Calling $p_j(T;i)$ $(i,j=1,-1)$ the conditional probabilities of j at time T, when $p_i=1$ at $T=0$, they are given by

$$\left.\begin{array}{l} p_1(T;\pm 1) \\ p_{-1}(T;\mp 1) \end{array}\right\} = \pm\frac{e^{-\eta T}}{2}+\frac{\eta/2}{\eta+2}e^{-(\eta+2)T}+\frac{1}{\eta+2} \ , \tag{6}$$

with the asymptotic values $p_{\pm 1}=1/(2+\eta)$. Identifying k with the asymptotic value $p_{\pm 1}(\infty)$ of (6), it is easily seen that the ratio η coincides with k'/k. Hence, use of experimental data of Fig. 3 for λ allows assignment of an η value to be put in the above kinematics.

Calling $x=x(t)$, $x'=x(t+\tau)$, the correlation function of the dynamical process is defined as the ensemble average over the joint probability distribution $p(x)p(x|x')$ of the two events, that is,

$$R(t,t+\tau)=\int dx\,dx'\,xx'p(x)p(x|x') \ , \tag{7}$$

and the averaged correlation function can be written as

$$\overline{R}(\tau)=\lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}R(t,t+\tau)\,dt \ . \tag{8}$$

As said above, the motions within each of three subspaces ± 1, 0 can be taken as decorrelated from the jumps, as well as decorrelated from one another. Therefore, the time average yields either the correlation functions $\langle V_i(t)V_i(t+\tau)\rangle$ $(i=\pm 1,0)$, or just $\langle V_i\rangle\langle V_j\rangle$ for $i\neq j$, while the probabilities $p(x)$, $p(x|x')$ reduce to the previously defined jump probabilities p_i and $p_j(\tau;i)$; hence

$$\overline{R}(\tau)=\sum_i\langle V_i(t)V_i(t+\tau)\rangle p_ip_i(\tau;i) + \sum_{i\neq j}\overline{V}_i\overline{V}_jp_ip_j(\tau;i) \ . \tag{9}$$

Neglecting the oscillating terms of $\langle V_i(t)V_j(t+\tau)\rangle$ which contribute to the frequency $\frac{1}{3}$, and besides a zero-frequency component, the low-frequency spectrum is made, in general, of two Lorentzians, plus a background corresponding to the fast mixing (with a rate γ) within the transient region, that is,

$$S(\omega)=(\langle V_1\rangle-\langle V_{-1}\rangle)^2\frac{a}{\omega^2+a^2} +(\langle V_1\rangle+\langle V_{-1}\rangle)^2\frac{a}{\omega^2+(a+2b)^2}+\frac{\langle V_0^2\rangle}{\gamma}\frac{b}{a+2b} \ . \tag{10}$$

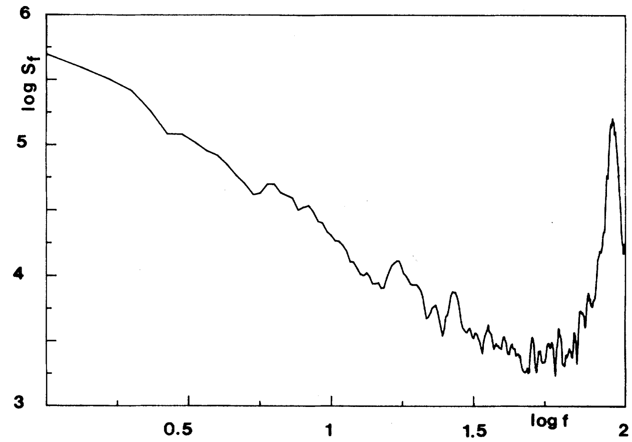

FIG. 4. Power spectrum for the driven Duffing oscillator $\ddot{x}+0.154\dot{x}-x+4x^3=0.114\cos(1.22t)$ with spectral density of added noise 6×10^{-6}. The peak on the right corresponds to $f=1.22/14\pi$ and comes from the two period-7 attractors.

For $\langle V_1\rangle=\pm\langle V_{-1}\rangle$ there is just one Lorentzian; for $\langle V_{-1}\rangle=0$, the two Lorentzians are weighted inversely proportional to their width. This latter case is particularly appealing insofar as a sum of a large number of Lorentzians weighted this way gives a spectrum which goes as $1/f$.

The considerations which have led to Eq. (5) implied a symmetry of the two attractors. If, in particular, $\langle V_1\rangle = \pm\langle V_{-1}\rangle$, the low-frequency spectrum reduces to a single Lorentzian; this has been confirmed by fitting the spectra of Fig. 2(b) with single Lorentzians plus a background with an accuracy of 1%. If, however, the two attractors have different exchange rates a_1,a_2 and b_1,b_2, there is no accidental cancellation of one Lorentzian. This is qualitatively confirmed by a numerical integration of the Duffing equation [Ref. 1, Eq. (1)], where the shortness of the transient regime is compensated for by the simultaneous existence of four attractors: two period-4 attractors and two period-7 attractors. The corresponding spectrum (Fig. 4) shows that here the power-law region extends one decade more than in Fig. 2(b).

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[1]F. T. Arecchi and F. Lisi, Phys. Rev. Lett. 49, 94 (1982). The slopes in the spectra of this paper must be multiplied by 2, for a miscalibration of the spectrum analyzer.

[2]F. T. Arecchi, R. Meucci, G. Puccioni, and J. Tredicce, Phys. Rev. Lett. 49, 1217 (1982).

[3](a) Y. Aizawa, Prog. Theor. Phys. 68, 64 (1982); (b) T. Geisel and J. Nierwetberg, Phys. Rev. Lett. 48, 7 (1982).

[4]E. Ben Jacob, I. Goldhirsch, Y. Imry, and S. Fishman, Phys. Rev. Lett. 49, 1599 (1982).

[5]M. R. Beasley, D. D'Humieres, and B. A. Huberman, Phys. Rev.

Lett. 50, 1328 (1983).

[6]C. Grebogi, E. Ott, and J. A. Yorke, Phys. Rev. Lett. 49, 1507 (1982).

[7]Similar spectra were obtained by P. Manneville, J. Phys. (Paris) 41, 1235 (1980), with a one-dimensional discontinuous map. The $1/f$ divergence is there due to a particular reinjection mechanism in the laminar phase, which excludes the region at left of the unstable fixed point. Such a limitation makes it hard to use that map to mimic physical processes, as stressed by the same author in his conclusions.

[8]I. Gumowski and C. Mira, *Dynamique Chaotique* (Cepadues, Toulouse, 1980).

[9]The immediate basin of attraction is defined (Ref. 8) as the neighborhood of an attractor within which the distance from the latter shrinks monotonically.