LOW-FREQUENCY HOPPING PHENOMENA IN A NONLINEAR SYSTEM WITH MANY ATTRACTORS

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We report experimental data on the power spectrum of a chaotic system (an electronic Duffing oscillator). The low-frequency portion of the spectrum is associated with noise-induced hopping among many basins of attraction.

The main routes to chaos in nonlinear dynamical systems explored thus far [1,2] involve the de-stabilization of an attractor [1] until it becomes strange or the sudden appearance of a strange attractor [2]. However the phase space of even the simplest nonlinear system with $n \geqslant 3$ degrees of freedom usually contains many independent basins of attraction, that is, manifolds of initial conditions leading asymptotically to different attractors. Addition of a small amount of noise causes hopping from one attractor to another, associated with the appearance of a low-frequency spectrum. The low-frequency spectrum corresponds to time tails in the correlation functions which are secular with respect to the internal correlation time within each attractor. Such an effect was first observed in an electronic Duffing oscillator [3] and later in a modulated CO_2 laser [4]. A model explanation was given in terms of a one-dimensional cubic map [5] allowing for two attractors plus wandering over a repulsive region. Low-frequency tails have also been reported in connection with slow motions over different subregions of a single attractor [6]. This is a deterministic diffusion process [7,8], which does not require additional noise, insofar as there are no transitions over different attractors.

In this communication we report a series of experimental data relative to the hopping regions of the oscillator of ref. [3]. The equation for the system is

$$\ddot{x} + \gamma\dot{x} - x + 4x^3 = A \cos \omega t + \xi(t) , \qquad (1)$$

where the dot denotes derivative with respect to the dimensionless time $\tau = \omega_0 t$, $\omega_0/2\pi$ is the resonant frequency of the Duffing oscillator in the linear region (660 Hz in our system); $\gamma = 0.154$; the dynamical range of x is of the order of 1 V. The relevant values of A and ω range between 0.55 and 2.3 V and 440 and 660 Hz, respectively. $\xi(t)$ is a stochastic gaussian process with zero average and correlation time much faster than any other system time, accounting for additive noise (either applied or intrinsic to the oscillator).

In fig. 1 we report the relevant features of our system in the parameter space (A, ω). Below the line L, as we change one of the two control parameters, we obtain the Feigenbaum sequence of subharmonic bifurcations for a single attractor confined in one potential valley, as reported in ref. [3]. Along the line L, the two attractors confined in the two potential valleys have orbits sufficiently extended to overcome the potential barrier for a small extra-amount of external excitation. Indeed we have measured in our system an r.m.s. noise of 600 μV and this is sufficient, along L, to allow hopping between the two attractors. In order to avoid introduction of a third control parameter, we keep constant this noise level and adjust only A and ω.

In fig. 2 we report the $(x, \dot{x})$ projections of the phase space trajectories, and the associated power spectrum in a log–log scale fitted by a power law $f^{-\alpha}$ in its low-frequency region. At variance with the model of ref. [5], in this oscillator the jump from one attractor to the other is sudden, without a relevant residence time within a repulsive region. Hence, until the

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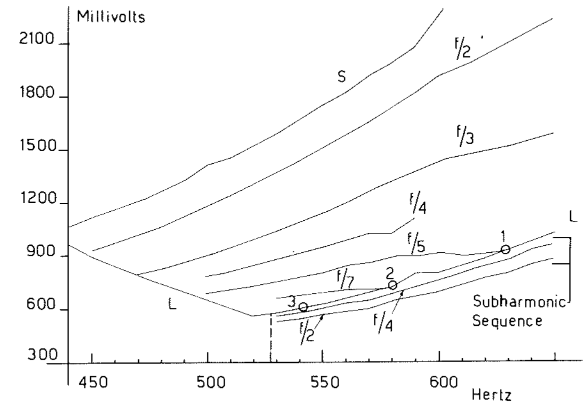


Fig. 1. Parameter space for the Duffing oscillator, showing the position of stable attractors. The subharmonic ratio is indicated for each attractor. Line L separates attractors confined in one valley from those going over two valleys. Points 1, 2, 3 show regions of coexistence of more than two attractors. Between lines L and S the behaviour is chaotic except for the narrow windows of stability of attractors, denoted by the lines (not all of them have been reported). From line S above there is the $f/1$ attractor over two valleys. From line L below there are only stable attractors within one valley. At the right of the dashed line, for increasing A, there is a Feigenbaum sequence of subharmonic bifurcation.

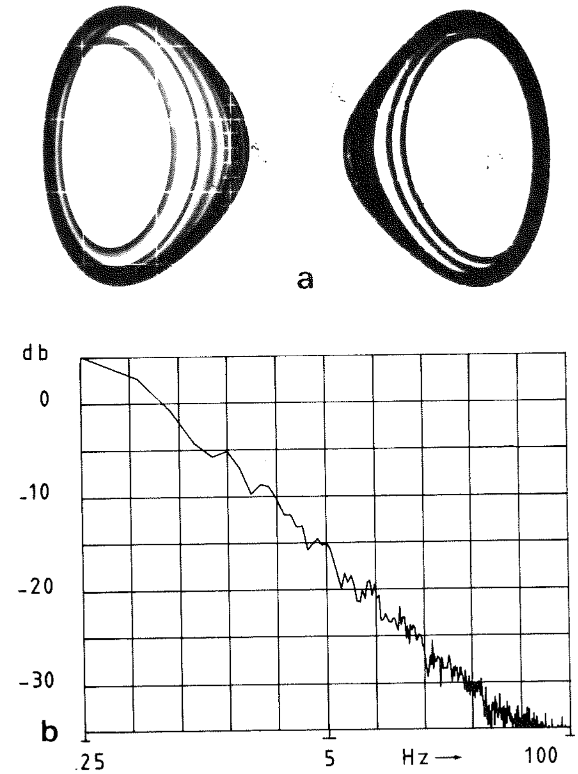


Fig. 2. (a) Phase-space plots of two symmetric attractors. (b) Low-frequency spectrum of jumps between the two attractors. Slope = 1.94; correlation coefficient = 0.73.

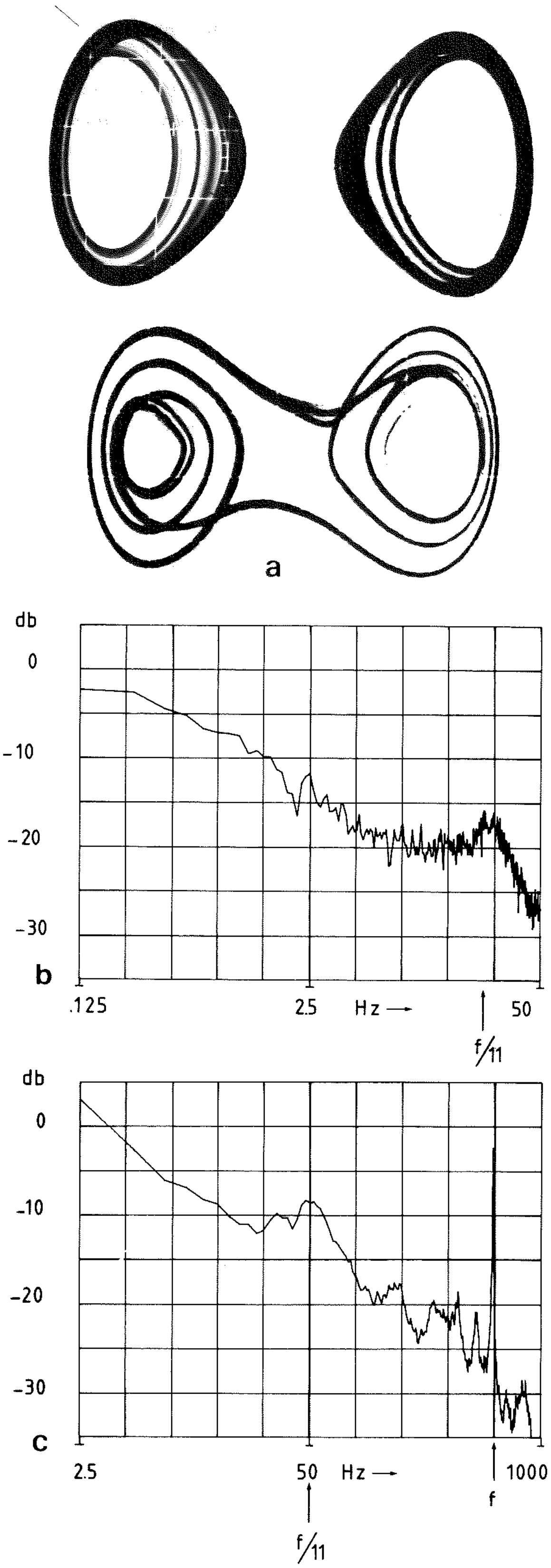
hopping is confined over only two objects, we must expect a Lorenz spectrum [5,9]. This is indeed the case shown in fig. 2b. It is interesting to notice from fig. 1 that the circles are regions of simultaneous overlaps of three attractors, namely the two symmetric ones discussed above plus a third one going over the two potential valleys.

In fig. 3 we show the three attractors and the associated power spectrum. The slope of the log–log plot is $\alpha \approx 1.28$. Fig. 3c shows how the jump spectrum continues under the high-frequency spectrum associated with the motion within each attractor.

In order to better discriminate between the jump spectrum and the chaotic motion within the (already strange) attractors, we have slightly changed the modulation amplitude A in order to isolate a stable (jumpless) motion either in one of the $f/2$ attractors, or on the $f/11$ attractor. The corresponding power spectra are given in fig. 4, and it is easy to subtract these contributions from fig. 3c in order to isolate the pure jump spectrum, not only in the low-frequency range (as in fig. 3b) but also underneath the high-frequency features (from $f/11$ to $f/2$).

In the first report on these jump spectra [3] the similarity to the $1/f$ noise observed in many physical systems was stressed. Here we have shown the change in the slope from $a = 2$ to $a = 1.28$ as we go from 2 to 3 attractors. Using a kinetic model to describe the jumps [5], we should expect N lorentzians when there are $N + 1$ attractors so that the slope 1.28 seems to fit a region arising from the sum of two lorentzians with suitable weights.

We have tried to extend this point of view to more than 3 simultaneous attractors, but for a fixed noise we did not obtain a substantially smaller exponent in the case of 4 attractors. A tentative explanation is the following. It has been known for a long time that a convenient way to build a $1/\omega$ spectrum, suitably cut off to keep its frequency integral finite, is to consider it as the superposition of a large number of lorentzians with time constants τ distributed with a density pro-



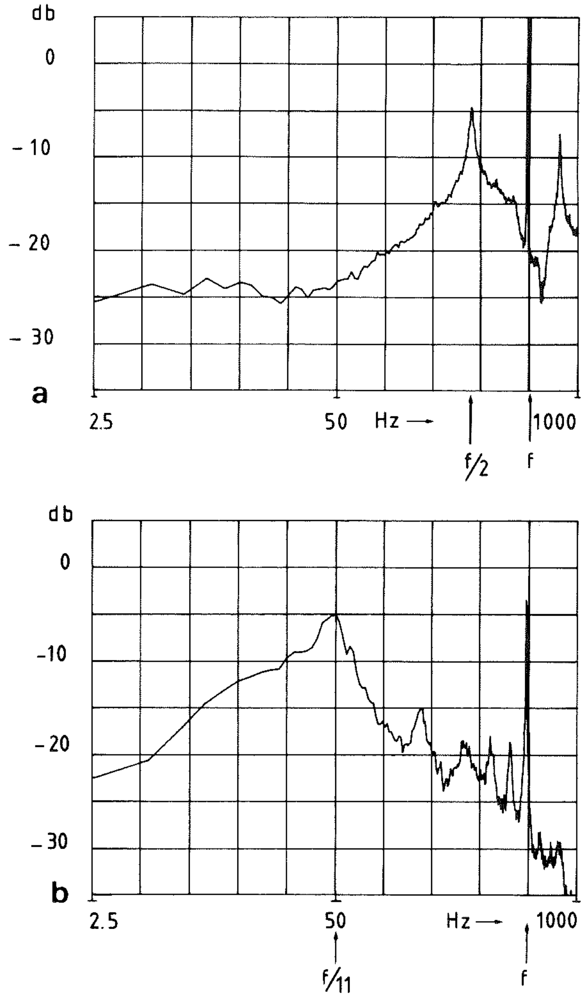


Fig. 4. Power spectrum of the separate motion for the: (a) $f/2$ attractor, (b) $f/11$ attractor.

portional to $1/\tau$ over a range (τ_1, τ_2). Indeed the resulting spectrum is

$$S(\omega) = \int_{\tau_1}^{\tau_2} \frac{\tau}{1+\omega^2\tau^2} \frac{1}{\tau}\, d\tau = \left.\frac{\tan^{-1}\omega\tau}{\omega}\right|_{\tau_1}^{\tau_2} . \qquad (2)$$

If τ_2/τ_1 is a large ratio, then the spectrum is $1/\omega$ over a corresponding large range. Many models have been

◀ Fig. 3. (a) Phase-space plots of three coexisting attractors ($f/2, f/2, f/11$). (b) Low-frequency spectrum of jumps between the three attractors. Slope = 1.28; correlation coefficient = 0.82. (c) High-frequency spectrum including motion within the attractors.

provided to justify the distribution of rate constants, but they appear built "ad hoc" for a specific situation. A very general explanation has been recently provided by Montroll and Shlesinger [10]. Whenever an event is conditioned by a sequence of previous ones, assuming the probability per unit time of each step independent of the others, the probability of the final event is the product of the probabilities of the individual preliminary events

$$P = p_1 p_2 \dots p_N , \qquad (3a)$$

and

$$\log P = \log p_1 + \log p_2 + \dots \log p_N . \qquad (3b)$$

When the individual distribution of the lof p_i values satisfy certain weak conditions the central limit theorem is applicable, so that the density of $\log P$ is the normal distribution. Now, a variable x obeying a log-normal distribution has a density $g(x) \sim 1/x$ over a wide range. Application of this argument to the time constant $\tau = P^{-1}$ of an event conditioned by a chain of previous events, yields a $1/\tau$ distribution to be put in eq. (2). Even though very general, the above argument imposes a strong requirement. In the case of jump phenomena among many attractors, we should look for a situation where: (i) there is a leakage from each attractor to a next one, and the last one can be reached only via a unique chain, and not via many parallel ways; (ii) the manifold of initial conditions feeds mainly a first attractor, and all the other ones have negligible basins of attraction.

These conditions are not satisfied by the attractors of the driven Duffing system. That is why, even when we sum over many lorentzians, they do not show up with the suitable weight implied by eq. (2).

To conclude, we have shown experimental evidence of low-frequency tails in the power spectrum corresponding to noise-induced jumps among different basins of attraction. The slope of these tails in a log–log plot varies between 1.7 and 1.24 depending on the number of attractors and on the way they couple to one another – that is, on geometrical details of the phase space. We have reported for comparison a conceptual argument on which an exact slope 1 is based, that is, the existence of a sequential rule of transitions like a logic and, to show that in a generic phase the jumps among attractors do not follow that sequential rule.

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