Acta Physica Austriaca 57, 119–126 (1985)

Acta Physica Austriaca

Stable Self-Pulsation and Chaos in Laser with Injected Signal*

By

F. T. Arecchi, G. Lippi, G. Puccioni, and J. Tredicce
Istituto Nazionale di Ottica, Firenze, Italy

With 5 Figures

(Received June 6, 1984)

Summary

A CO_2 frequency-locked laser to an external reference is shown to have oscillatory and eventually chaotic behavior. A linear stability analysis shows that the steady state can be achieved only when the injected CO_2 laser field is resonant with the internal one. When detuning is accounted for, the effective laser frequency is either the internal or the external one. The first case is obtained in the limit of zero injected field amplitude A, the second one requires A strong enough to completely slave the system. If A has intermediate values, the output intensity is unstable because of competition between the two modes of operation. Through more and more complicate intensity oscillation patterns, the system is shown numerically to yield oscillations of increasing complexity, up to chaos, this being reached via intermittency.

Chaotic output behavior has been recently described in a model of laser with a constant injected field [1], the relevant variables being the field, population inversion and polarization for which five real equations were used to describe the dynamics. However, in many experimentally realizable cases, all those variables do not remain equally important for the dynamics because of differences in their relaxation rates.

Here, we report results for a reduced set of equations suitable for modelling a CO_2 laser with injected signal including detunings of the cavity resonant frequency (ω_c) and the injected frequency (ω_1) from the atomic resonant frequency (ω_0). After adiabatic elimination of the polarization because of its rapid relaxation rate ($\gamma_\perp \sim 10^8\ \mathrm{sec}^{-1}$ for CO_2)

---

* Contributed paper presented at the Seminar on Fundamentals of Quantum Optics in Obergurgl, Austria, 1984.

the coupled equations for the complex field amplitude $\tilde{a}$ (normalized so that $|\tilde{a}|^2$ is the photon number) and the population inversion S_3 (normalized so that its maximum is the total molecular population) are

$$\dot{\tilde{a}} = -k\tilde{a}(1 + i\theta) + \frac{g^2 \tilde{a} S_3}{\gamma_\perp(1 - i\delta)} + kBe^{-i\Delta t},$$

$$\dot{S}_3 = -\gamma_\parallel(S_3 - S_0) - \frac{4g^2|\tilde{a}|^2 S_3}{\gamma_\perp(1 + \delta^2)}, \tag{1}$$

where k, $\gamma_\parallel$ are the loss rate for the field and the population inversion, respectively, $g^2 = \omega\mu^2/\hbar\epsilon_0 V$ is the coupling constant (μ = dipole matrix element, V = cavity volume), S_0 is the pump rate, B the amplitude of the external field,

$$\Delta = (\omega_1 - \omega'), \qquad \theta = (\omega_c - \omega')/k, \qquad \delta = (\omega' - \omega_0)/\gamma_\perp,$$

ω' being the reference frequency where we expect the system to oscillate. Notice that ω' is undefined, and it has to be deduced by consistency arguments. Immediately we see that the steady state solution $\dot{\tilde{a}} = \dot{S}_3 = 0$ represents the situation where $\Delta = 0$ and thus it can only be achieved for $\omega' = \omega_1$. Thus all steady state solutions represent frequency locking of the laser with the injected signal. Hence without loss of generality, we take the natural reference frequency $\omega' = \omega_1$. For convenience of computation, we normalize the complex amplitude $\tilde{a}$ to the saturation field $(\gamma_\perp\gamma_\parallel)^{1/2}/2g$ and the population inversion S_3 to the threshold value $k\gamma_\perp/g^2$, for zero detuning, that is, we introduce

$$x = 2g\,\mathrm{Re}(\tilde{a})/\sqrt{\gamma_\perp\gamma_\parallel},$$

$$y = 2g\,\mathrm{Im}(\tilde{a})/\sqrt{\gamma_\perp\gamma_\parallel},$$

$$z = g^2 S_3/k\gamma_\perp. \tag{2}$$

Now x, y and z are numbers of the order of unity for regular behavior, and Eqs. (1) can be written as [2]

$$\dot{x} = \left(\frac{z}{1+\delta^2} - 1\right)x + \left(\theta - \frac{\delta z}{1+\delta^2}\right)y + A,$$

$$\dot{y} = -\left(\theta - \frac{\delta z}{1+\delta^2}\right)x + \left(\frac{z}{1+\delta^2} - 1\right)y,$$

$$\frac{k}{\gamma_\parallel}\dot{z} = -\frac{z}{1+\delta^2}(x^2 + y^2) - z + z_0, \tag{3}$$

where $A = 2gB/\sqrt{\gamma_\perp\gamma_\parallel}$, and the dot denotes the derivation with respect to the adimensional time $\tau = kt$.

If $A = 0$ the laser oscillates at a frequency ω_L imposed by the frequency pulling formula for a homogeneous laser line (as it is the case for CO_2 lasers) .

$$\frac{\omega_c - \omega_L}{k} = \frac{\omega_L - \omega_0}{\gamma_\perp}. \tag{4}$$

Another important factor to be taken into account is the laser stability without the injected signal which yields the eigenvalues

$$\lambda = -\frac{\gamma_\parallel z_0}{2} \pm \sqrt{(\gamma_\parallel^2 z_0^2/4) + 2k\gamma_\parallel(1 - z_0)}. \tag{5}$$

For the CO_2 case where $z_0 \ll \gamma_\parallel \ll k$, λ becomes

$$\lambda \cong -\frac{\gamma_\parallel z_0}{2} \pm i\sqrt{2k\gamma_\parallel(z_0 - 1)}. \tag{6}$$

Thus, in presence of a perturbation the laser reacts with a damped oscillation lasting approximately $2/\gamma_\parallel z_0$.

The existence of a real frequency $(2\gamma_\parallel k)^{1/2}$ suggests that a resonant excitation around that frequency could destabilize the nonlinear dynamical system. Following this philosophy, in a previous work [3] we have obtained chaotic behavior for an external driving around that frequency. Hence here we guess that a beat frequency around $(\gamma_\parallel k)^{1/2}$ between injected and internal fields may induce a resonant destabilization.

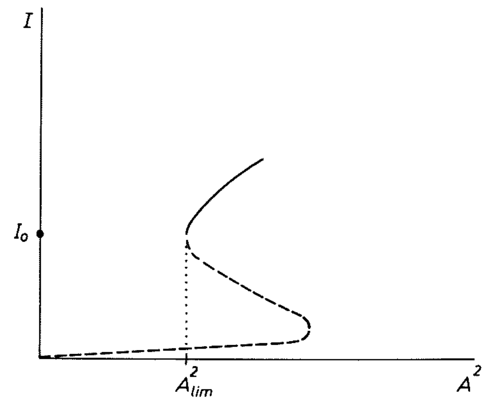

Fig. 1. Intensity of the external field (A^2) versus the intensity of the system ($x^2 + y^2$) for $\omega_1 \neq \omega_L$. The dashed line indicates unstable regions. Note that if $A < A_{\text{lim}}$ any steady state solution is stable and if $A = 0$ (no injected signal) the intensity ($x^2 + y^2$) is I_0

Considering now $A \neq 0$, a steady state analysis of Eqs. (3) gives

$$\bar{z} = \frac{z_0}{1 + \frac{x^2 + y^2}{1 + \delta^2}} \tag{7}$$

and

$$A^2 = \left\{ \left[\frac{\bar{z}}{1 + \delta^2} - 1 \right]^2 + \left[\theta - \frac{\delta \bar{z}}{1 + \delta^2} \right]^2 \right\} (x^2 + y^2). \tag{8}$$

In Fig. 1 we show a plot of A^2 versus $(x^2 + y^2)$ with ω_1 different from ω_L. We can distinguish qualitatively different regions. The first one is

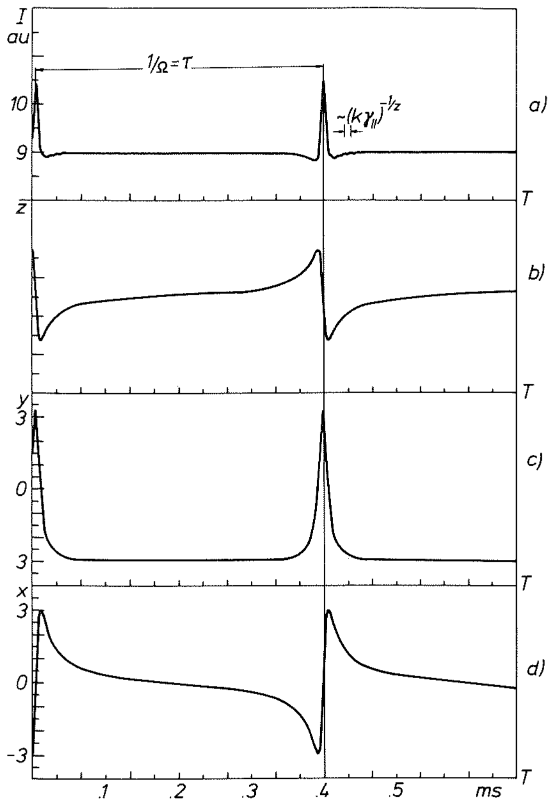

Fig. 2. *a* Intensity versus time for $A = 0.0137$, $\delta = 0.0124$, $\theta = 0.017$. *b* Population inversion versus time. Note that it grows continuously until it decays rapidly transferring the energy to the field. *c* Imaginary part of the field y versus time. *d* Real part of the field x versus time. Note that in *c* and *d* there is a rapid change in the signal of x and y indicating a change in phase of 2π

for A bigger than A_{lim}, where the steady state solution is stable and the system (laser + injected signal) oscillates at the frequency ω_1. In the second (for $A = 0$) the laser oscillates at ω_L. For A different from zero and smaller than A_{lim}, no steady state solution is possible and there is a competition between the external field oscillating at a frequency ω_1 and the laser itself which is (at least initially) at ω_L. Fixing $\omega_1 - \omega_L$ = constant and reducing A to zero the output intensity oscillates at a frequency which is proportional to $(\omega_1 - \omega_L)$. However, more generally, when $A \lesssim A_{\text{lim}}$ we see two different frequencies giving a stable pulsation of the system (Fig. 2). Those frequencies arise from different processes. One of them is similar to a Q-switching process. In this case it is clear that the field amplitude assumes a state which is practically in quadrature with the external field, and it is constant during a long time. Thus we conclude that the external field A gives a contribution only to increment the population inversion which grows until it transfers energy to the inphase part of the field (x) causing a peak in the graph of the intensity.

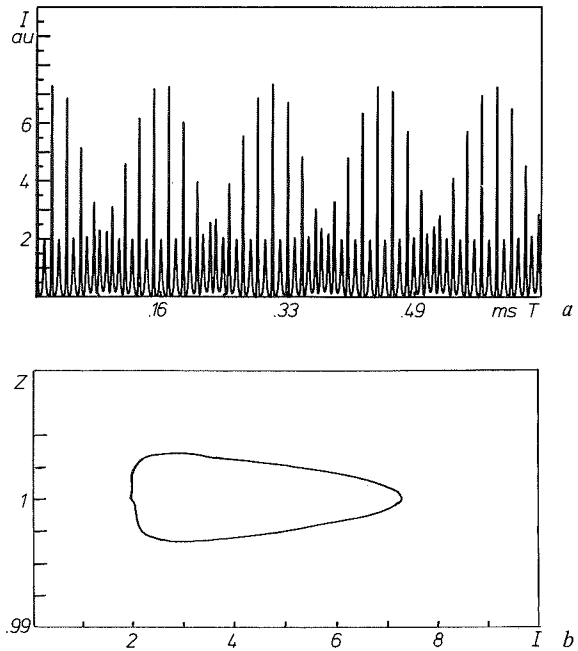

Fig. 3. *a* Intensity of the laser radiation versus time for the control parameter $A = 0.021$, $\delta = 0.01$, $\theta = 0.025$. *b* Poincaré section in the plane $\Delta - I$ shows a continuous line, corresponding to the intersection of a torus with a plane. The continuity of the section is evidence that the two frequencies are incommensurate

This process can be understood also by saying that the external field A tries to lock the frequency of the system at its own frequency but it is not strong enough to maintain this situation stable. This explains our results which show that the frequency of those pulsations (Ω) increases as A decreases, and the maximum value is reached for $A \to 0$; so the maximum value is proportional to $(\omega_1 - \omega_L)$. The second frequency corresponds to the damped oscillations for the laser without injected signal. This can be clearly seen in Fig. 2 where these oscillations are damped when the operation frequency of the system is near to ω_1 (locking situation). This frequency grows in amplitude when the nearly phase-locked condition is lost.

As $(\omega_1 - \omega_L)$ is much less than $\sqrt{k\gamma_{\|}(z_0 - 1)}$ there is no resonant excitation and the behavior of the system is periodic. However, increasing $\omega_1 - \omega_L$ just until the frequency Ω becomes of the order of the damped

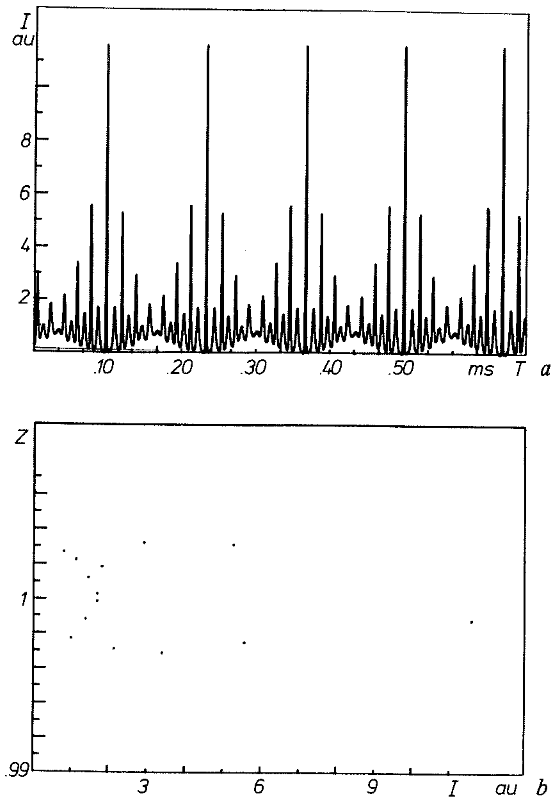

Fig. 4. For $A = 0.02295$ the two frequencies lock at a ratio $\Omega_1/\Omega_2 = n_1/n_2$ with n_1, n_2 integers. This is shown by the finite number of points in the Poincaré section b

oscillations, a resonant destabilization is induced. In such a case the time behavior of the intensity (Fig. 3a) is made of two sequences of spikes H-L-H-L (H = high, $I \sim 70$; L = low, $I \sim 20$) and it appears also a third low frequency which is incommensurate with the others as it appears from the continuous torus section in the Poincaré map (Fig. 3b). Increasing the value of A the frequencies lock at a commensurate value and we observe a discrete number of points corresponding to the two-torus (Fig. 4). A further increase in the parameter A destabilizes the lower frequency by intermittency as is shown in Fig. 5 and the system goes eventually to a chaotic regime.

In conclusion, we have shown a transition to chaos via intermittency for the typical parameters of a CO_2 laser by injecting a steady non-

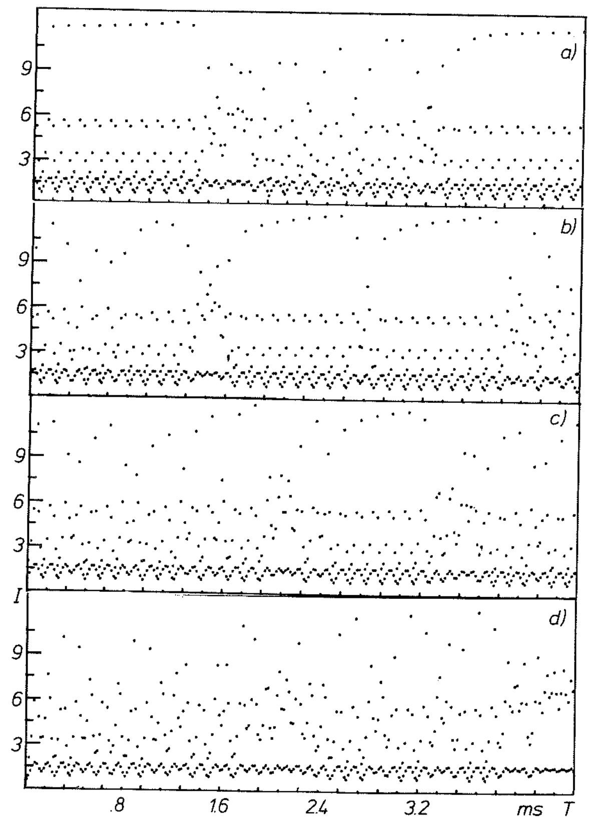

Fig. 5. For A bigger than 0.02295 the system enters in an intermittency region. Increasing A from a to d the laminar period decreases and the system goes eventually to a chaotic regime

modulated signal. Furthermore our calculations confirm our heuristic picture of a damped nonlinear system which goes into steady oscillations because of a resonant excitation around $\sqrt{k\gamma_{\|}(z_0 - 1)}$. Searching the origin of the frequencies we have been able to give an explanation of the different processes which are relevant for the dynamics.

The authors thank N. B. Abraham for helpful discussions.

References

[1] Lugiato, L. A., Narducci, L. M., Bandy, D. K., Pennise, C. A.: Opt. Commun. **46**, 64 (1983).
[2] Arecchi, F. T., Lippi, G. L., Puccioni, G., Tredicce, J.: In: Coherence and Quantum Optics, V (Mandel, L., Wolf, E., eds.). New York: Plenum 1984.
[3] Arecchi, F. T., Meucci, R., Puccioni, G., Tredicce, J.: Phys. Rev. Lett. **49**, 1217 (1982).

Printed in Austria