SCALING OF FIRST PASSAGE TIMES FOR NOISE INDUCED CRISES

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Nonlinear dissipative systems can have many simultaneously coexisting basins of attraction (generalized multistability – GM [1,2]. This situation can be destabilized by change of the control parameters, merging two independent attractors into a single one via an intermediate region which is only sporadically visited near the transition. The associated dynamics implies a low-frequency tail (deterministic diffusion [3–5]). Vice versa, when the above coexistence is stable, application of external noise may induce jumps between two otherwise disjoint regions of phase space.

For some simple systems we study the two different phenomenologies near their borderline, giving a scaling relation for the escape rates in terms of control parameter and noise amplitude.

Clear evidence of generalized multistability was first shown in an electronic oscillator and then in a modulated laser system [1]. In both cases, besides the qualitative appearance of different attractors in phase space, there was a low-frequency spectral component due to noise-induced jumps among different attractors. Both measurements, however, might be considered as experimental artifacts. In fact, there is evidence of single attractors made of two sub-regions with unfrequent passages from one to the other (see e.g. the Lorenz attractor [5] for $\sigma = 10$ and $r = 28$). In such a case, the low-frequency tail corresponds to the sporadic passages, and it does not require added noise (deterministic diffusion [3]). As a matter of fact, power spectra do not allow one to discriminate between the two phenomena.

A clear analysis of GM has been so far limited to an oversimplified model, namely a cubic recursive map, allowing for two simultaneous attractors [2]. Here we present a detailed numerical study of a differential system, that is, a forced Duffing oscillator with a double-well potential, whose equation is

$$\ddot{x} + \gamma\dot{x} - x + 4x^3 = A\cos\omega t. \qquad (1)$$

Numerical solution of eq. (1), for $\gamma = 0.154$ and for different choices of the parameters A, ω, yields a particularly interesting region of the parameter space displayed in fig. 1. There we have reported the boundaries of the regions of existence of some attractors, corresponding to subharmonics of order from 4 to 7, as denoted by the associated numbers. The meaning of the two lines C and D may be explained with reference to the two valleys of the potential of eq. (1). Line C is a borderline above which the motion is no longer confined in one single valley. Below line C, there is a manifold of lines approximately parallel to it, corresponding to a sequence of subharmonic bifurcations, with mutual distances ruled by Feigenbaum's universal rate δ. These have been omitted in fig. 1 for clarity reasons. They have already been observed experimentally (see fig. 1 of ref. [6]) in an electronic oscillator ruled by eq. (1), which is, however, affected

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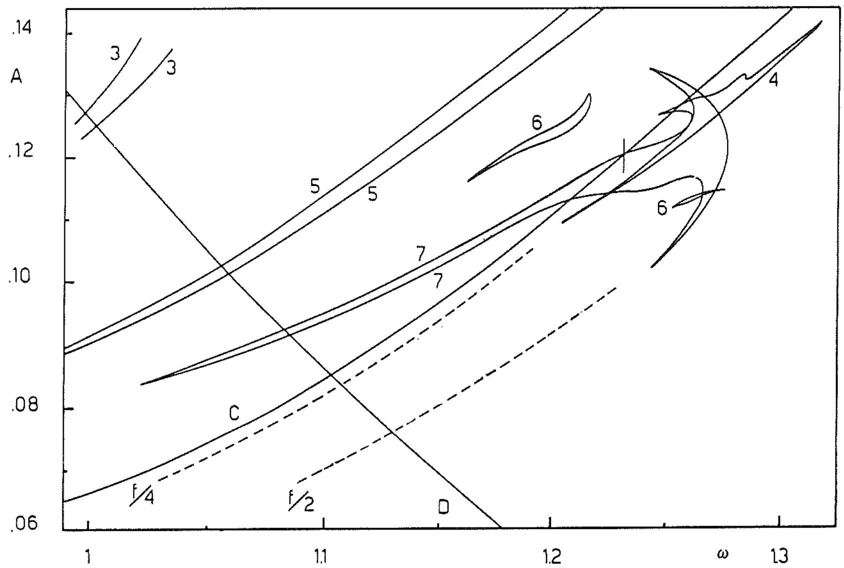

Fig. 1. Phase diagram of Duffing equation showing the type of solutions for each pair of driving parameters (A, ω). The borderlines are the frontiers of parameter regions corresponding to ordered motion, and the associated numbers denote the periodicity. Curves C and D show the upper limit (in the amplitude A) for the stability region of solutions confined in one valley. The vertical bar at ω = 1.22 indicates the region where the escape times have been measured.

by too large a noise to be able to display the other interesting details reported in fig. 1.

On the left of line D, there is a small limit cycle confined in one valley. This limit cycle does not undergo subharmonic bifurcations, but it dies by intermittency on the line D. In particular, in the triangular region below the two lines C and D there is coexistence of the small limit cycle with one of the abovementioned Feigenbaum chain.

We measure the mean escape time from the period-7 attractor versus the amplitude σ of an applied noise in the region denoted by a vertical bar in fig. 1. Specifically, at each integration step (which is 10^{-2} the forcing period) we shift $\dot{x}$ by a random number sorted from a rectangular distribution with zero mean and r.m.s. σ.

Fig. 2 shows log T versus $1/\sigma$ for different values (A_1, A_2, A_3, A_4) of the driving amplitude A at fixed frequency ω. $A_3 \approx A_c$ is the parameter at which the period-7 undergoes crisis [7], that is, the representative point "escapes" from the attractor after an infinite time, in the absence of noise. This is equivalent to the definition of crisis [7], i.e. A_c is the value for which the attractor of period-7 collides with the unstable period-7 solution.

For $A_4 > A_c$ even a zero noise ($1/\sigma \to \infty$) yields a finite escape time as it is evident from the corresponding saturated plot. For $A < A_c$, $T \to \infty$ as $\sigma \to 0$, the faster the smaller is A, as shown by comparison between A_1 and A_2. The finite escape time across a bounded region for $A > A_c$ corresponds to the deterministic (noise-free) diffusion discussed elsewhere. On the contrary, for $A < A_c$, the attractor is structurally stable, therefore, the phase point cannot escape, unless we apply external noise. This is the phenomenon of noise-induced jumps already reported experimentally [1] and simulated with a one-dimensional map [2]. Notice that, both for $A > A_c$ and $A < A_c$, around the crisis region the large escape times give low-frequency power spectra which are qualitatively similar. The essential difference is that for $A < A_c$ no jumps occur in the absence of noise.

It is apparent from fig. 1 that one can have the same T for different A's, adjusting the noise amplitude. This may be expressed in terms of a scaling relation as given for other chaotic scenarios [8–10]. For simplicity we refer to the logistic map

$$x_{n+1} = \mu x_n (1 - x_n), \qquad (2)$$

with μ around the crisis value $\mu_c = 4$. The invariant

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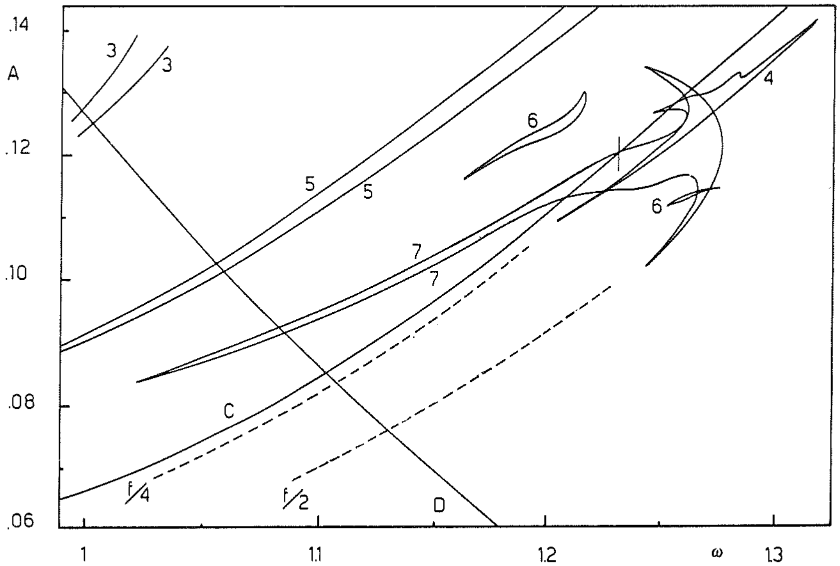

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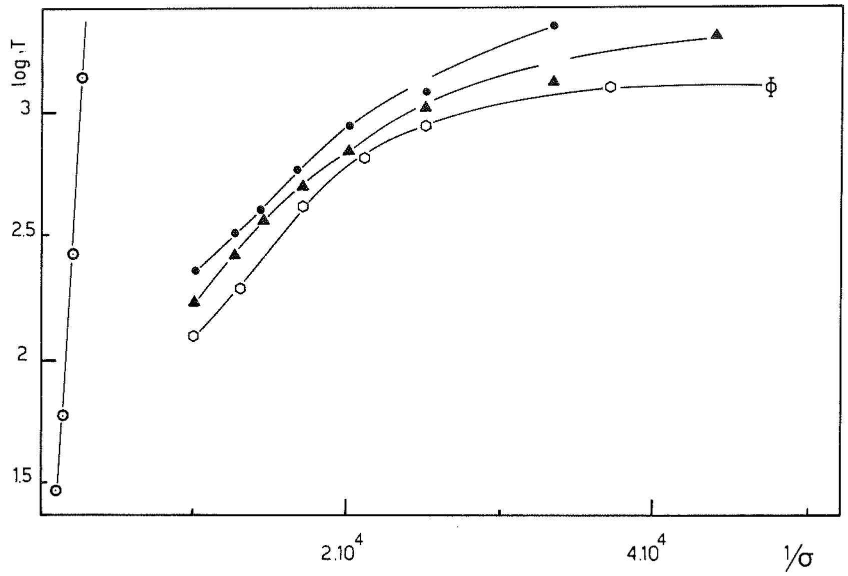

Fig. 2. The mean escape time from the period-7 region versus the inverse of the noise amplitude. All the curves refer to the same frequency, but with different A's. Namely the symbols ⊙ ● ▲ ○ represent respectively: A = 0.1165, 0.117275, 0.117280, 0.117285. The error bar is drawn only for one measurement since it is always the same.

density for the noise-free map is given by [11]

$$\rho_c(x) = [\pi\sqrt{x(1-x)}]^{-1}. \qquad (3)$$

The divergence of ρ_c in $x = 1$ is due to the flat behaviour of the map around the pre-image point $x = 1/2$.

We first evaluate the mean escape rate R from the right for $\mu \sim \mu_c$. If either a small noise signal is added or μ is set slightly above μ_c, there is no longer an invariant density; however, we can reasonably assume that the distribution of points relaxes to a pseudo-invariant $\rho(x)$ on a time scale much shorter than the mean escape time. In particular, for $\mu < \mu_c$ and without noise, $\rho(x)$ may be a highly singular distribution, however, an additional random term smooths out almost all of the peaks over a continuous $\rho(x)$, whose rightmost part is a scaled version of the corresponding part of $\rho_c(x)$, that is, $\rho(x) \approx (\pi\sqrt{1-\epsilon-x})^{-1}$ with the asymptote shifted from $x = 1$ to $x = 1 - \epsilon$ ($\epsilon = 1 - \frac{1}{4}\mu$).

When applying a gaussian noise of r.m.s. σ, the escape rate R at the point x is equal to the area of the gaussian centered at x, outside the segment (0, 1). Hence, for $\epsilon \ll 1$, we have the escape rate from right given by ($y = 1 - x - \epsilon$)

$$R(\epsilon,\sigma) = \frac{1}{\pi}\int_0^\infty \frac{\mathrm{d}y}{\sqrt{y}} \int_y^\infty (2\pi\sigma^2)^{-1/2} \exp[-(z+\epsilon)^2/2\sigma^2]\,\mathrm{d}z. \qquad (4)$$

Integrating by parts we have

$$R(\epsilon,\sigma) = \frac{2}{\pi}\int_0^\infty \sqrt{y}\,(2\pi\sigma^2)^{-1/2} \exp[-(y+\sigma)^2/2\sigma^2]\,\mathrm{d}y,$$

and hence [12]

$$R(\epsilon,\sigma) = \pi^{-1}\sqrt{\sigma/2}\,\exp(-\epsilon^2/4\sigma^2) D_{-3/2}(\epsilon/\sigma), \qquad (5)$$

where D is the parabolic cylinder function.

It is readily seen that the function $R(\epsilon, \sigma)$ satisfies the general scaling law

$$R(\epsilon,\sigma) = \sigma^\alpha F(\epsilon/\sigma^\beta), \qquad (6)$$

with the exponents $\alpha = 1/2$ and $\beta = 1$. Similar scaling laws have already been given for the period-doubling cascade [8,9] and for intermittency [10], but with different exponents, as well as different physical meaning for R, as discussed later in the conclusion.

At the crisis ($\epsilon = 0$) R scales as the square root of σ, namely,

$$R(0, \sigma) = [\Gamma(3/4)/\pi^{3/2}]\, 2^{1/4}\sigma^{1/2}. \qquad (7)$$

Below the crisis ($\epsilon > 0$) and for $\epsilon \gg \sigma$, we have the asymptotic relation

$$R(\epsilon, \sigma) = (\sqrt{2}/\pi)(\sigma^2/\epsilon^{3/2}) \exp(-\epsilon^2/2\sigma^2). \qquad (8)$$

Above the crisis ($\epsilon < 0$) we can escape even without noise: indeed the asymptotic expansion of eq. (5) in the limit of large ϵ/σ yields $R \sim (2/\pi)|\epsilon|^{1/2}$ as already given in ref. [7].

To evaluate the left escape rate L, we start again from the crisis density which has the same divergence at $x = 0$ as at $x = 1$. However, while the divergence at $x = 1$ is provided by injection from an inner region, the divergence at $x = 0$ is a straightforward consequence of the density around its pre-image $x = 1$ and hence strongly affected even by a small noise.

We show now that L is much smaller (around 10%) than R at $\mu = \mu_c$ and even less important for $\mu < \mu_c$ for noise-induced jumps.

For $\mu > \mu_c$ the noiseless escape always occurs at the right (image of points around the maximum). Numerical studies show that, both for gaussian and rectangular noise, the left contribution is around 10% (namely 13% for the former and 11.5% for the latter) of the right one. Hence, we consider sufficient to develop a handy argument for L based on the application of a rectangular noise of r.m.s. σ. At the crisis, the density $\rho(x) = (\pi\sqrt{1-x})^{-1}$ around $x = 1$, once convoluted with the rectangular noise, yields a distribution

$$P(x) = (\pi\sigma\sqrt{3})^{-1}(1 + \sqrt{3}\sigma - x)^{1/2},$$

$$1 - \sqrt{3}\sigma < x < 1 + \sqrt{3}\sigma, \qquad (9)$$

whose area on the right of 1 directly gives the escape rate $\bar{R}$

$$\bar{R}(0, \sigma) = (2/3^{3/4})\, \sigma^{1/2}. \qquad (10)$$

Incidentally, we can observe that the rectangular noise yields the same scaling relation as the gaussian one [see eq. (7)] with the only difference of a multiplicative factor 7% larger in eq. (10).

By applying the recursive map (2), we get a first approximation of the pseudo invariant $\rho(x)$ around $x = 0$

$$\rho(x) \approx (8\sqrt{3}\pi\sigma)^{-1}(x + 4\sqrt{3}\sigma)^{1/2},$$

$$0 < x < 4\sqrt{3}\,\sigma. \qquad (11)$$

Now, a convolution of eq. (11) with the noise gives a distribution $P(x)$ whose area on the left of 0 yields a first order contribution to L. Furthermore, by applying again the map, since $x = 0$ is an unstable fixed point, the remaining area of P is partially shifted away. Hence, we get a second-order contribution to L by taking the convolution with the noise. The procedure is rapidly convergent: indeed, each contribution is about 7 times smaller than the previous one. The sum of the first three terms is

$$\bar{L} = 3.155 \times 10^{-2}\sigma^{1/2}, \qquad (12)$$

yielding a theoretical prediction of the ratio $\bar{R}/\bar{L} \sim 8.85$. This is in agreement with the numerical estimate value 8.69 ± 0.09. Now, by summing up the left and right contribution we are able to evaluate the global mean escape time for the interval (0, 1). A good agreement between the numerical and theoretical results is shown in fig. 3.

In conclusion, we have analysed the effect of an external noise in a dynamical system near a crisis. We have described the noise-induced jumps by means of mean escape times from the attractor, An analytic expression for a scaling law has been found showing a strict analogy with the period-doubling and the intermittency phenomenon. While in the period doubling [8,9] the relevant quantity is the Lyapunov exponent and in the intermittency [10] is the laminar length,

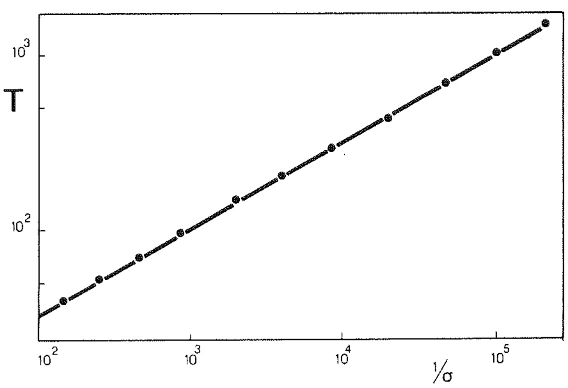

Fig. 3. The mean escape time from the interval (0, 1) versus the inverse of noise amplitude for the logistic map at $\mu = 4$ and with a gaussian noise. The straight line shows the theoretical slope.

here the major role is played by the mean first-passage time.

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