DETERMINISTIC CHAOS IN LASER WITH INJECTED SIGNAL

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Dynamic behaviour of an homogeneously broadened laser with injected signal is analyzed for a model in which the polarization is adiabatically eliminated. Detuning between the atomic resonant frequency, the cavity resonance and the frequency of the external signal is considered. We show that a transition to chaos via intermittency is possible for parameters appropriate for CO_2 lasers.

Recently chaotic behaviour has been observed in a few single mode laser systems either working with a homogeneous [1,2] or inhomogeneous [3] gain line. As pointed out in ref. [1] the beauty of a single mode laser with a homogeneous line is that the relevant instabilities can be modelled with the small number of equations resembling the Lorenz model. This was originally stressed by Haken [4]. However two conditions were necessary, one for excitation, that pump be of the order of ten times the threshold value, and one for the time scales, namely, that the cavity loss rate be larger than the sum of population and polarization decay rates. As a matter of fact, these conditions have not been realized so far in any working system. Indeed, the Maxwell-Bloch equations for a complex single mode field amplitude, a complex polarization and a population inversion amount to five nonlinear equations which are more than sufficient to yield deterministic chaos for suitable parameter values. In particular when the field is tuned to the center of the gain line, the field and polarization become real and the three resulting equations are fully equivalent to the Lorenz equations.

Real atoms and molecules are constrained by time scale considerations which reduce the phase space dimensionality down to the point where instabilities are quenched [5]. In the most familiar atomic gas lasers (He-Ne (0.633 μm), Ar^+, Kr^+) and in dye lasers, population and polarization decay much faster than the field, so the corresponding equations can be solved at steady state (adiabatic elimination procedure [6]) and the dynamics are governed by a single field equation. We call these systems Class A. In some other systems (Class B, as e.g. ruby, Nd and CO_2 lasers) the population decays slowly, so that the dynamics is described by two coupled rate equations which are still insufficient to yield dynamic instabilities. We call Class C systems those ones for which the three decay rates for the polarization, population and field are of the same order of magnitude. Thus far, the only practical examples of Class C systems are the FIR lasers [7] however no instabilities have been reported for simple systems.

In order to increase by at least 1 the number of the degrees of freedom of Class B systems we must either

(i) make the systems non autonomous, by a time dependent parameter, or

(ii) increase the number of lasing modes, or

(iii) increase the number of independent gain packets, that is, using an inhomogeneous line.

Point (i) was shown in ref. [1]. Point (ii) has been shown for two-counter-rotating modes in a ring cavity [8–10] [‡1,2]. Point (iii) was treated in experimental

‡1 There is an extensive literature on Class B NdYAG bidirectional ring laser systems. See for example ref. [8], and references therein.

‡2 Oscillatory instabilities in a ring cavity have been shown previously for a Class A (dye) laser [10]. Such systems are described by only two coupled equations, thus appearance of deterministic chaos is forbidden.

and theoretical papers [3,11], without however a simple analysis in terms of a closed set of equations. In such a case, due to the large number of available packets, distinction in Class A and B is immaterial. The same can be said for combinations of point (ii) and (iii) (multimode, inhomogeneous line) [12].

We have recently reported a much simpler approach to chaos [13], by injection of an external signal into a homogeneous line, Class B, laser, with a detuning between external and internal frequencies, so that both real and imaginary parts of the field amplitudes are relevant dynamical variables ‡3,4. In this communication we give a detailed treatment of the transition to chaos in single mode Class B lasers with an injected unmodulated signal. The Maxwell-Bloch equations describing N homogeneously broadened 2-level atoms coupled to a single mode field are [6]

$$\dot{E} = -\mathrm{i}\omega_c E - gP - kE + kE_0 \exp(-\mathrm{i}\omega_1 t)\,,$$

$$\dot{P} = -\mathrm{i}\omega_0 P - gE\,\Delta N - \gamma_\perp P\,, \qquad (1)$$

$$\Delta\dot{N} = 2g(EP^* + E^*P) - \gamma_\parallel(\Delta N - \Delta N_0)\,,$$

where E the complex field amplitude, P is the complex polarization, ΔN is the population inversion and k, $\gamma_\perp$, $\gamma_\parallel$ are their respective loss rates. E_0 is the injected field, ΔN_0 is the unsaturated population inversion, ω_0, ω_c and ω_1, are the atomic, cavity and external field frequencies respectively and

$$g^2 = \omega\mu^2/\hbar\epsilon_0 V \qquad (2)$$

is the coupling constant in terms of the dipole moment μ and the cavity volume V. The normalization is such that E^2 is the photon number and P and ΔN are measured in units of N.

This model applies to a Class B laser, such as, e.g.,

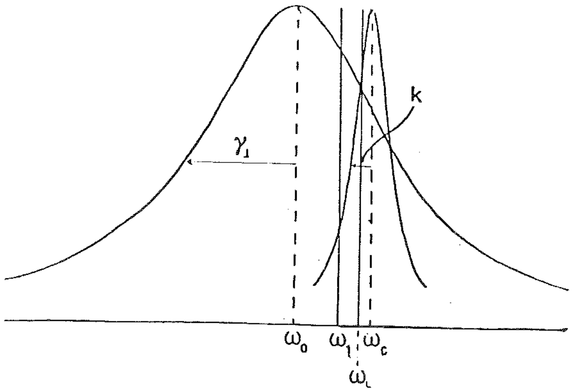


Fig. 1. Qualitative plot of the frequency relations among atomic line (homogeneous width $\gamma_\perp$) centered at ω_0, cavity line (width k) centered at ω_c, and external field centered at ω_1. The choice of the frequency position is such that the injected signal acts as an perturbation around the position of the internal laser frequency, imposed by the pulling condition.

a CO_2 laser medium where the pressure broadening provides a homogeneous gain line. A CO_2 laser system has a population relaxation time ($1/\gamma_\parallel = 10^{-3}$ s) much longer than the dipole decay time ($1/\gamma_\perp = 10^{-8}$ s), hence the single mode dynamics is described by the coupling between two degrees of freedom (rate equations). Introduction of the external field whose frequency ω_1 is detuned with respect to ω_0 and ω_c provides the three degrees of freedom necessary for the onset of chaos.

Relying on the condition $\gamma_\perp \gg k, \gamma_\parallel$, we can solve the second of eqs. (1) at steady state and substitute the function $P(E, D)$ into the other two (adiabatic elimination) ‡5. Furthermore, we rewrite the remaining equations in a frame rotating with a generic angular frequency ω' that is,

$$\dot{E} = -kE(1 + \mathrm{i}\theta) + (g^2/\gamma_\perp)\,E\,\Delta N/(1 + \mathrm{i}\delta) + kE_0 \exp(-\mathrm{i}\alpha k t) \qquad (3)$$

$$\Delta\dot{N} = -\gamma_\parallel(\Delta N - \Delta N_0) - (4g^2/\gamma_\perp)|E|^2\Delta N/(1 + \delta^2)\,,$$

where

$$\alpha = (\omega_1 - \omega')/k, \quad \theta = (\omega_c - \omega')/k, \quad \delta = (\omega' - \omega_0)/\gamma_\perp\,. \qquad (4)$$

‡3 A more complicated variant of this experiment was proposed for a Class A system in ref. [14]. In such a case, besides the external injection is also necessary to introduce an explicit time dependence either in the external field or in the pump.

‡4 For a Class C system, chaos induced by an injected signal has been seen in ref. [15]. Having already three dynamical variables they have enough degrees of freedom to release a condition, which is instead essential for us, namely that the atomic and cavity frequencies be detuned. Furthermore, addition of an external field makes the "bad cavity" condition unnecessary. This approach suffers however the practical limit of all Class C systems.

‡5 The adiabatic elimination here used might seem too simple minded with respect to the general conditions regulating this approximation (see e.g. ref. [16]). However it has been checked that introducing our numerical values into the complete set of Maxwell Bloch equations, one obtains the same results as in our simplified scheme [17].

Here by inspection we see that steady state solution ($\dot{E} = \Delta\dot{N} = 0$) represents the situation where $\alpha = 0$ and it can only be achieved by assuming an operating frequency ω' equal to the frequency of the injected signal. Thus all steady state solutions correspond frequency locking of the laser to the injected signal. Stability analyses will tell us if this locked condition can be maintained in the presence of small perturbation. Hence, without loss of generality, we take as a natural reference frequency the external one: $\omega' = \omega_1$ and we write the complex amplitude E as $E_r + iE_i$. For convenience of computation, we normalize the field to the saturation field $(\gamma_\perp\gamma_\parallel)^{1/2}/2g$ and the population inversion to its threshold value $k\gamma_\perp/g^2$ for zero detuning. This way the dimensionless variables

$$x = E_r 2g/(\gamma_\perp\gamma_\parallel)^{1/2}, \quad y = E_i 2g/(\gamma_\perp\gamma_\parallel)^{1/2}$$

$$z = g^2 \Delta N/k\gamma_\perp\,, \tag{5}$$

are of the order of 1 for "regular" behaviour (before the onset of spiking instabilities) and eqs. (2) can be written as [14]

$$\dot{x} = [z/(1+\delta^2) - 1]x + [\theta + \delta z/(1+\delta^2)]y + A\,,$$

$$\dot{y} = -[\theta + \delta z/(1+\delta^2)]x + [z/(1+\delta^2) - 1]y\,,$$

$$(k/\gamma_\parallel)\dot{z} = -z(x^2+y^2)/(1+\delta^2) - z + z_0\,, \tag{6}$$

where now the dot denotes the derivation with respect to the dimensionless time $\tau = kt$, and $A = 2gE_0/(\gamma_\perp\gamma_\parallel)^{1/2}$.

To further understand the behaviour of the system we note that if $A = 0$, the laser, above threshold, oscillates at a frequency ω_L different from ω_1, where ω_L is determined by the periodic boundary conditions imposed by the single mode of operation being considered. This frequency is given by the frequency pulling formula for a homogeneous laser line

$$(\omega_c - \omega_L)/k = (\omega_L - \omega_0)/\gamma_\perp\,. \tag{7}$$

A linear stability analysis without injected signal yields the eigenvalues

$$\lambda = -\gamma_\parallel z_0/2 \pm [\gamma_\parallel^2 z_0^2/4 + 2k\gamma_\parallel(1 - z_0)]^{1/2}\,. \tag{8}$$

For the CO_2 case where $\gamma_\parallel \ll k$, λ becomes

$$\lambda = \lambda_r + i\lambda_i \simeq -\gamma z_0/2 \pm i[2k\gamma(z_0 - 1)]^{1/2}\,, \tag{8'}$$

Thus, in the presence of a perturbation, the laser reacts with damped oscillation lasting approximately $1/\lambda_r$. The existence of an oscillatory response with frequency λ_i suggests that a resonant excitation around that frequency might destabilize this nonlinear system. Following this philosophy, in previous work [1] we have obtained chaotic behaviour by modulating the cavity losses at approximately that frequency. Here we conjecture that a difference frequency between injected and internal fields around λ_i may induce a resonant destabilization, even for very small A amplitudes. In solving numerically for solutions of eqs. (5) we have several control parameters to be fixed. In the results presented here we have chosen the cavity eigenfrequency $\omega/2\pi$ at 1.75 MHz above the atomic frequency and the pump ratio with respect to the threshold value $z_0 = 10$. This will immediately fix the frequency position ω of the internal laser for $A = 0$ and the photon number (normalized to the saturation photon number $n_s = \gamma_\perp\gamma_\parallel/4g^2$ as $(n_0/n_s) = x_0^2 + y_0^2 = z_0 - 1 = 9$. Choosing $\omega_1/2\pi$ to be 130 kHz below $\omega_L/2\pi$ ($\delta = -0.01, \theta = 0.025$) and for different external amplitudes A, we obtain the behaviour of the photon number $n = x^2 + y^2$ and of the instantaneous frequency

$$\omega_{\text{inst.}} = \omega_1 - \dot{\phi}\,, \tag{9}$$

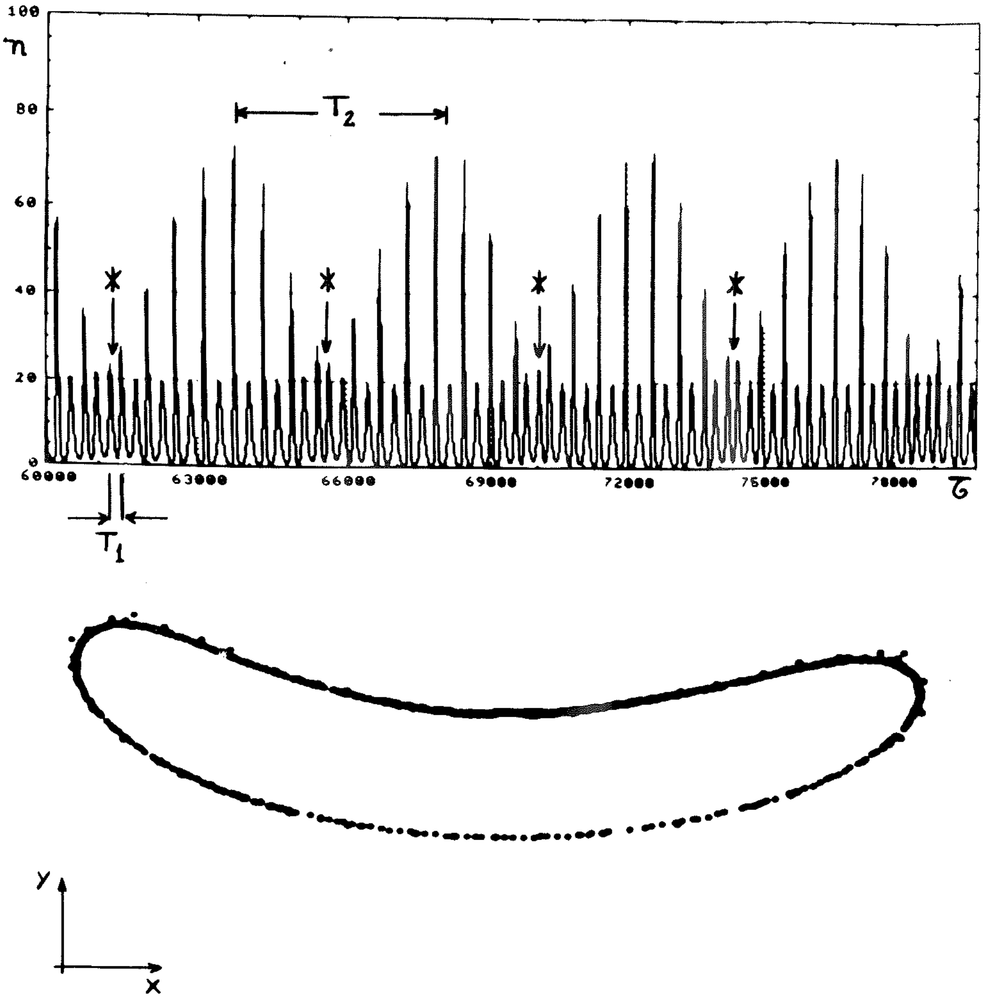
where

$$\phi = \arg(x + iy)\,,$$

and

$$\dot{\phi} = d\phi/d(kt) = (\dot{y}x - \dot{x}y)/(x^2+y^2)\,.$$

Leaving to a successive work a detailed analysis of ϕ, here we present a set of data for $n(t)$ and for the Poincaré phase space sections (x, y), taken for $\dot{n} = 0$. It will not be surprising that in the presence of a very small injected signal ($|A|^2 \ll n_0$) the operation of the laser as viewed from the ω_1 reference frame is a periodic oscillation of frequency proportional to the difference frequency $(\omega_1 - \omega_L)$ between the unperturbed laser and the injected signal which is nearly the eigenvalue frequency resulting from the linearization of eq. (6). Thus we expect to find the steady state solution (laser locked to the injected signal) "unstable" with respect to such pulsations for small values of A. In fig. 2a the time behaviour is made of two sequences of spikes H–L–H–L (H = high, $n \sim 70$, L = low, $n \sim 20$). We notice that there are strong



(a)

(b)

Fig. 2(a). Intensity $n = x^2 + y^2$ of the laser radiation versus the normalized time $\tau = kt$ for the control parameter $A = 0.021$. (b) The Poincaré section in the plane x–y shows a continuous line, corresponding to the intersection of a torus with a plane. The continuity of the section is evidence that the two frequencies are incommensurate.

fluctuations around the average value $n_0 \sim 9$ due to an injected signal just 2% of n_0. We call the "high frequency" $\Omega_1 = 2\pi/T_1$ the inverse of the separation time T_1 between adjacent peaks and the "low frequency" $\Omega_2 = 2\pi/T_2$ that one associated with the H peak separation T_2. It is immediately seen that Ω_1 is of the order of 5×10^4 s^{-1} (near to $[2(z_0-1)(k\gamma)]^{1/2} \simeq 7.1 \times 10^4$ s^{-1}) and Ω_2 is approximately 17 times smaller. The two frequencies are not commensurate as is shown by the continuous torus section in the Poincaré map (fig. 2b). As a result around the points denoted as (*) in fig. 2a, the sequence H–L–H–L is altered, the local spacing differs from T_1 and this local perturbation decay over a time of the order of $\gamma_\parallel^{-1} \sim 10^{-3}$ s. This confirms our heuristic picture of a damped nonlinear system which goes into steady oscillations because of a resonant excitation around $\lambda_i \simeq (k\gamma_\parallel)^{1/2}$. For increasing values of the external field, the system does not go to the chaotic behaviour via quasi-periodicity, but the two frequencies first lock at a commensurate value. In the Poincaré map we observe a discrete number of points corresponding to the locking on a two-torus (fig. 3). The number of points is equal to the denominator of the ratio $\Omega_1/$

(a)

(b)

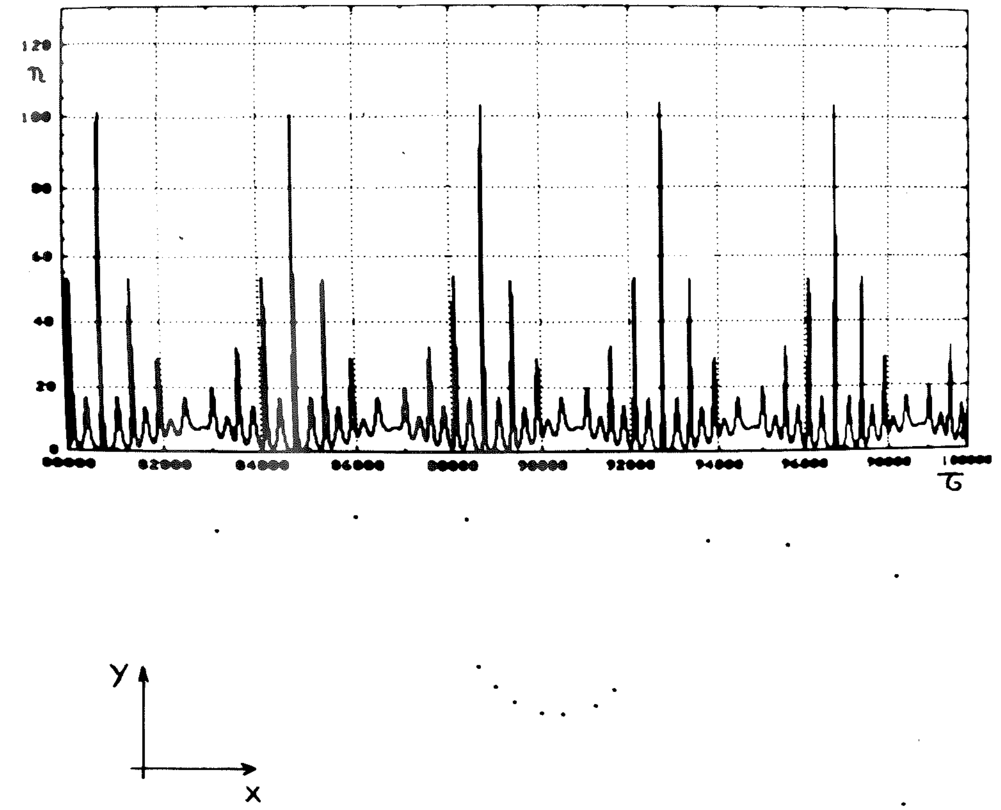


Fig. 3. For A = 0.2295 the two frequencies lock at a ratio n_1/n_2 (n_1, n_2 integers) where n_2 = 14. This is shown by finite number of points (namely, 14) in the Poincaré section (b).

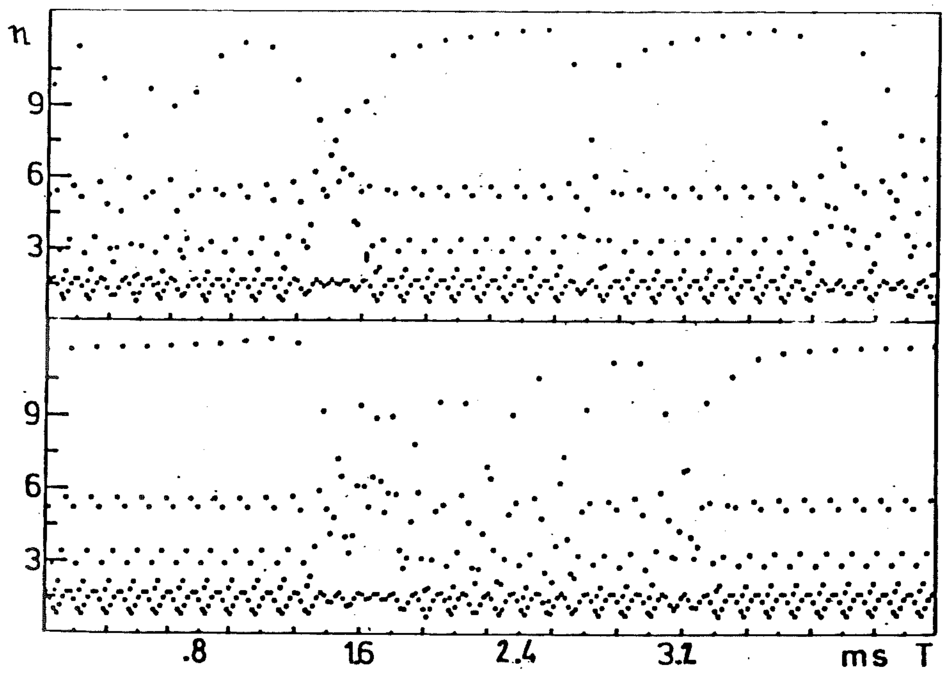


Fig. 4a.

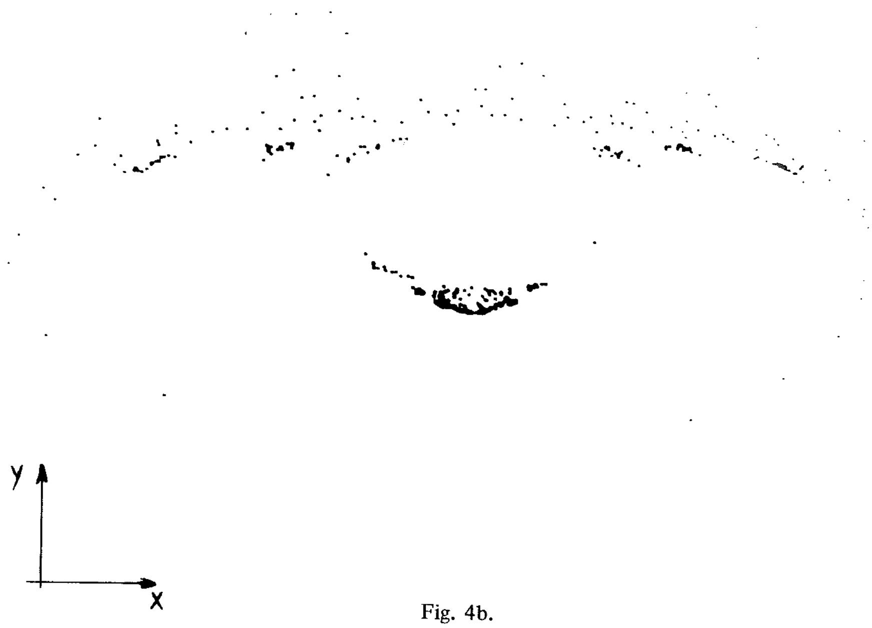


Fig. 4b.

Fig. 4. For $A \geqslant 0.02295$ the frequencies Ω_1, Ω_2 unlock and the system enters an intermittency region. This is shown clearly in the time domain (limited to the sequence of peaks). The corresponding Poincaré section shows a broadening around the points of the previous plot.

Fig. 5. For $A = 0.02297$ the system has become chaotic. Since the time plot would be irrelevant, we show the Poincaré section, which is organized in "sheets", each one broadened.

$\Omega_2 = n_1/n_2$ (n_1, n_2 integers) of the two frequencies. A further increase in the parameter A destabilizes the lower frequency by intermittency as is shown in fig. 4 and the system goes eventually to a chaotic regime (fig. 5). This route to chaos seems to be a physical realization of the idealized route to chaos described in terms of a map of the circle [18,19]. A described comparison between our numerical experiment and a map of the circle requires further work.

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