PHYSICAL REVIEW LETTERS

VOLUME 55 — 22 JULY 1985 — NUMBER 4

Measurement of the Formation and Evolution of a Strange Attractor in a Laser

G. P. Puccioni, A. Poggi, W. Gadomski,[a] J. R. Tredicce,[b] and F. T. Arecchi[c]
Istituto Nazionale di Ottica, I-50125 Firenze, Italy
(Received 25 April 1985)

We have measured the fractal dimensions and the Kolmogorov entropies of periodic and chaotic attractors for a CO_2 laser system with modulated losses. In particular, we find an increase in dimension near the accumulation point of the periodic cascade according to the Feigenbaum scaling law, besides the expected usual increase of the attractor dimension in the chaotic region. Numerical solutions of a theoretical model yield dimensions in quantitative agreement with the experiments, thus demonstrating a close match of experiment and theory for this physical system.

PACS numbers: 05.45.+b, 05.70.Ln, 42.50.+q

Over the past years, several authors[1,2] have treated the problem of distinguishing deterministic chaos from random noise. As one of the readily measurable characteristics of a strange attractor is its fractal dimension, a measurement of this quantity is an important test, not only to distinguish whether an observed broad power spectrum corresponds to deterministic chaotic behavior or just to noise, but also to characterize the strange attractor. Different algorithms have been proposed and applied to numerical results originated by return maps or nonlinear differential equations.[3,4] More recently similar calculations were performed on experimental data from several systems[5–7] including a laser.[8] However, a complete correlation between theory and experiment has been lacking thus far because reliable models are unusually complicated and detailed time-dependent results had not yet been generated.

Here we present an accurate experimental test of the fractal dimensionality of the attractor for a CO_2 laser with modulated losses. For the first time, the fractal dimension at the accumulation point of a Feigenbaum cascade is experimentally measured and, also for the first time, the experimental results can be compared with results from the theory of such a system.

The experimental setup was described previously,[9] and some subharmonic frequencies and eventually chaos were shown to exist. For the purpose of this Letter we remark that the stability of the experimental system has been greatly improved and a very stable $f/8$ subharmonic frequency and even an $f/10$ periodic window inside the chaotic region have been observed.[10] To give an idea of the reliability of the apparatus we report here a series of behaviors observed at 2% modulation depth for slight changes of the modulation frequency, controlled via a programmable synthesizer driven by a microprocessor. In the following sequence the number is the set frequency (in kilohertz), and then the relevant subharmonics are indicated: 191.290, $f/5$ and $f/4$; 191.313, $f/3$ and $f/4$; 191.320, $f/2$ and f; 191.324, f; 191.327, $f/2$ and $f/3$; 191.331, $f/3$; 191.337, $f/4$. This is just a sample from a much larger data collection.

As we keep the modulation frequency constant at 191.000 kHz and increase the modulation depth from 1% to 20%, the system passes through a period-doubling cascade up to the accumulation point and enters a fully chaotic region. The chosen frequency is close to the nonlinear laser resonance Ω introduced in Ref. 9. It depends on the damping rates k of the cavity and γ of the molecular population and on the relative amount A of pumping above threshold as $\Omega = [k\gamma(A-1)]^{1/2}$. As we scanned the frequency we found a narrow tongue of maximum sensitivity around Ω, where the laser destabilizes with the least amount of modulation.

The signal was digitized by a LeCroy transient recorder with 32 000 samples in memory. Setting the internal clock at 320 ns, we obtained approximately 16 points for each period of the fundamental frequency with eight-bit resolution. By synchronizing the sampling time to the external drive period we obtained a projection of the Poincaré section. The projection is onto a one-dimensional space (we measure only the intensity) independent of the other variables. In Fig. 1 we present the sections and the corresponding time series, respectively. The advantage of this signal processing is that we are able to analyze a high number of periods (32 000 maximum) with a single acquisition. Furthermore, it allows a much larger-bandwidth processing of narrow pulse sequences, which otherwise requires a high sampling rate with the related problems in data storing and processing. In Fig. 1, on the left-hand side, the bandwidth is 300 kHz, and on the right it is 100 MHz; indeed we can notice already in the $f/8$ plot a loss of resolution in the smaller peaks on the left-hand side.

We analyze digitized time sequences of the laser output intensity and reconstruct the attractors with an embedding technique.[3,5] Different procedures for the determination of the fractal dimension have been proposed[1-4] and here we follow the method described in Ref. 1.

If we define $N_n(\epsilon)$ as the number of vectors whose distance is smaller than ϵ, and if the embedding dimension n is large enough, then $N_n(\epsilon) \sim \epsilon^\nu$, where ν is a characteristic dimension of the attractor. In Figs. 2(a)–2(f) we plot $\log N_n(\epsilon)$ as a function of $\log\epsilon$ for a sequence of bifurcations $f/4$, $f/8$, and chaos. We will not discuss here the slope of the curves for small or high values of $\log\epsilon$, as it has been done by several authors,[5-9] but we limit our analysis to the regions where it remains constant over a wide region of $\log\epsilon$ and where it is independent of n, as it must be from theoretical predictions.

From inspection of Figs. 2(a) and 2(b) it is clear that the slope obtained for the $f/4$ subharmonic saturates at $\nu \cong 1$ in the time series and $\nu \cong 0$ in the Poincaré section. For the $f/8$ subharmonic ν is slightly above 1.5 [Figs. 2(c) and 2(d)]. This result, even though not readily understandable because the time signal still appears periodic, nevertheless agrees with the theoretical prediction for the dimension at the accumulation point (infinite periodicity) of the logistic map $(1.5376 < \nu < 1.5385)$. Indeed this dimension has been proven to be universal for those mappings for which the Feigenbaum scaling law holds.[11] We present here a heuristic interpretation based on our data. In our experimental system, the unavoidable noise yields a trajectory wandering over a nonzero

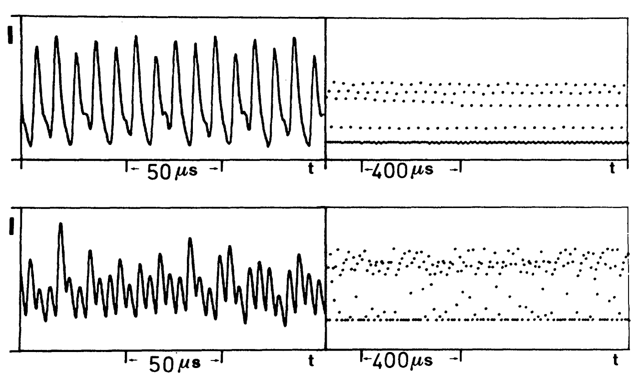

FIG. 1. Top: Laser intensity vs time for an $f/8$ subharmonic frequency, and corresponding stroboscopic intensity plot with the time interval between successive points equal to the period of the modulation frequency (191.000 kHz). Bottom: Laser intensity vs time and stroboscopic plot for chaotic behavior. The period of the modulation is 5.2 μs. We note on the left-hand side the loss of resolution due to the limited acquisition bandwidth. This drawback is absent on the right-hand side because of the huge increase in bandwidth.

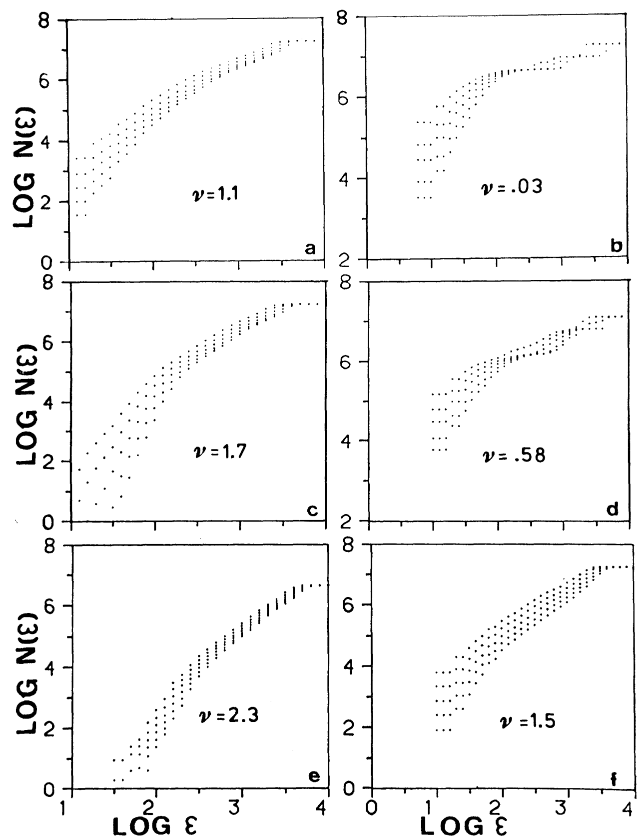

FIG. 2. Plots of $\log N_n(\epsilon)$ vs $\log\epsilon$ for different values of n calculated from the time series (left-hand panels) and from the stroboscopic sections (right-hand panels) for different subharmonic frequencies (a),(b) $f/4$; (c),(d) $f/8$; and (e),(f) chaotic behavior. All best-fit values of the slope ν are assumed to have an overall estimated error of ± 0.1. 6000 points were used. The embedding dimensions for all reported plots run from 5 to 9. Dimensions from 1 to 15 were tested.

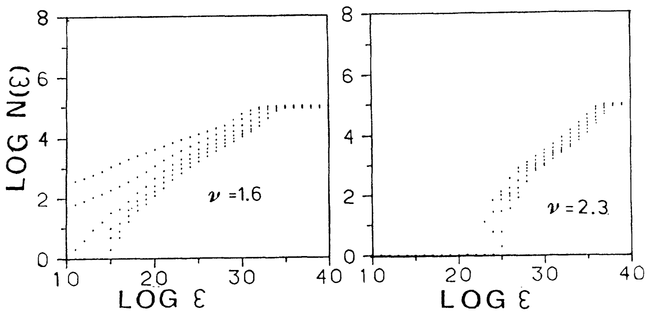

FIG. 3. Plots of $\log N(\epsilon)$ vs $\log \epsilon$ for different dimensions n obtained from the numerical integration of the model equations for two different cases, $f/8$ subharmonic (left-hand side) and chaos (right-hand side). 6000 points were used.

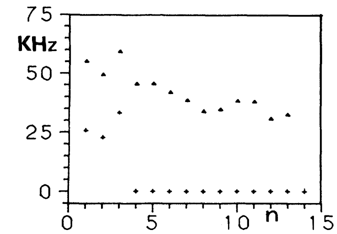

FIG. 4. Correlation entropy K_2 (in kilohertz) vs the embedding dimension for the $f/4$ (crosses) and the chaotic (triangles) attractor.

range of parameter values, thus "testing" nearby periodic attractors of the subharmonic sequence.[12] For infinite resolution, we would see for the stroboscopic data a staircase of horizontal plateaus each with zero slope, as it appears at higher embedding dimensions in Figs. 2(b) and 2(d). However, the finite resolution of the correlation measurements averages over adjacent steps, and thus provides the 0.58 slope, as it appears in Fig. 2(d). This is the first time that the dimension at the accumulation point of a Feigenbaum cascade has been measured in an experimental system.

When the system enters the chaotic region, the fractal dimension suddenly jumps to a higher value ($\nu = 2.4$) according to the general theory of strange attractors.[13]

The CO_2 laser system used here can be modeled by two first-order nonautonomous differential equations as described in Ref. 9. The time behavior of the intensity obtained by numerical integration of those rate equations was processed in the same manner as the experimental signal. Figure 3 shows the results obtained for an $f/8$ solution and a strange attractor. Again near the accumulation point $\nu \cong 1.5$. Direct comparison of Fig. 3 with Figs. 2(c) and 2(d) shows a good agreement between experiment and model.

It is important to stress that this agreement between theory and experiment is obtained with no floating parameters, but just our feeding the equations of Ref. 9 with the experimental values of the parameters, namely the following: the unmodulated cavity damping rate, $K_1 = 3\times 10^7$ sec^{-1}; the relaxation rate of induced dipoles, $\gamma_\perp = 10^8$ sec^{-1}; the collisional relaxation rate of population inversion (at the working pressure and discharge current), $\gamma_\parallel = 2.5\times 10^3$ sec^{-1}; the Einstein coupling coefficient of rate equations (for the definition, see Ref. 9), $G = 0.25\times 10^{-4}$ sec^{-1}; and the frequency and amplitude of loss modulation as in the experiment, that is, the frequency set at 191.000 kHz and $m = 2.0\%$ and 2.85%, respectively, for the left- and right-hand sides of Fig. 3.

The high regularity of the stroboscopic $N(\epsilon)$ plots for increasing of embedding dimension [Figs. 2(b), 2(d), and 2(f)] suggests the application of a method[1] which gives a lower estimate of the Kolmogorov entropy. In Fig. 4 we report the correlation entropy K_2 as defined in Ref. 1 versus the embedding dimension for the $f/4$ and for the chaotic attractor. We see that while $K_2 = 0$ for $f/4$, $K_2 \cong 35$ kHz for the chaotic attractor. As we have a single positive Lyapunov exponent and as the embedding time is 5.2 μs, we estimate that the half-loss of information corresponds to 3.8 periods of the modulation frequency.

In conclusion, our experiment has a new feature with respect to previous ones,[5-9] insofar as, for the first time, there is a strict quantitative correspondence between the experimental chaos measured in the laboratory and the theoretical chaos provided by the model, which thus confirms that the Maxwell-Bloch equations with adiabatic elimination of the polarization are valid for the modeling of our experimental system even in the chaotic region.

We thank P. Poggi, S. Mascalchi, and S. Acciai for technical assistance and N. B. Abraham, G. L. Oppo, S. Ciliberto, and A. Politi for helpful discussions. This research was partly sponsored by the Commission of the European Community under the experimental phase of European Community Stimulation Action (1983–1985).

---

[a]Permanent address: Department of Chemistry, Warsaw University, Warsaw, Poland.

[b]Present address: Department of Physics, Drexel University, Philadelphia, Pa. 19104.

[c]Also with the Physics Department, University of Firenze, Firenze, Italy.

[1]P. Grassberger and I. Procaccia, Phys. Rev. A **28**, 2591 (1983).

[2]J. D. Farmer, E. Ott, and J. Yorke, Physica (Amsterdam) **7D**, 153 (1983).

[3]P. Grassberger and I. Procaccia, Phys. Rev. Lett. **50**, 346 (1983).

[4]R. Badii and A. Politi, Phys. Rev. Lett. **52**, 1661 (1984).

[5]A. Brandstatter, J. Swift, H. Swinney, A. Wolf, J. D. Farmer, E. Jen, and J. Crutchfield, Phys. Rev. Lett. **51**, 1441 (1983).

[6]B. Malraison, P. Atten, P. Berge, and M. Dubois, J. Phys. (Paris), Lett. **44**, L897 (1983).

[7]M. Giglio, S. Musazzi, and U. Perini, Phys. Rev. Lett. **53**, 2402 (1984).

[8]A. M. Albano, J. Abounadi, T. H. Chyba, C. E. Searle, S. Young, R. S. Gioggia, and N. B. Abraham, J. Opt. Soc. Am. B **2**, 47 (1985).

[9]F. T. Arecchi, R. Meucci, G. P. Puccioni, and J. R. Tredicce, Phys. Rev. Lett. **49**, 1217 (1982).

[10]A. Poggi, G. P. Puccioni, W. Gadomski, J. R. Treddice, and F. T. Arecchi, in Proceedings of the Conference on Lasers and Electro-Optics, Baltimore, Maryland, 21–24 May 1985 (to be published).

[11]P. Grassberger, J. Stat. Phys. **26**, 173 (1981).

[12]J. P. Crutchfield, J. D. Farmer, and B. A. Huberman, Phys. Rep. **92**, 20 (1982).

[13]J. L. Kaplan and J. A. Yorke, in *Functional Differential Equations and Approximation of Fixed Points,* edited by H. O. Peitgen and H. O. Walther, Lecture Notes in Mathematics, Vol. 730 (Springer, Berlin, 1980), p. 228.