EUROPHYSICS LETTERS 1 January 1987

Europhys. Lett., 3 (1), pp. 5-10 (1987)

Noise-Induced Trapping at the Boundary between two Attractors: a Source of $1/f$ Spectra in Nonlinear Dynamics.

F. T. ARECCHI and A. CALIFANO (*)

Istituto Nazionale di Ottica,
Physics Department of the University - Florence, Italy

(received 30 April 1986; accepted in final form 1 September 1986)

PACS. 05.40. – Fluctuation phenomena, random processes, and Brownian motion.
PACS. 72.70. – Noise processes and phenomena.

Abstract. – We show that the power spectrum of a nonlinear dynamical system with more than one basin of attraction becomes $1/f$ in the presence of noise, provided the basin boundary be fractal. The result is numerically proved by a cellular automaton. This phenomenon resembles the random-random walk in a one-dimensional lattice, which undergoes a subdiffusive motion with the same type of spectra.

In this letter we show how addition of random noise in a nonlinear dynamical system with more than one attractor may lead to $1/f$ spectra, provided that the basin boundary be fractal. This shows that combining the features leading to deterministic chaos with a random noise is somewhat equivalent to a double randomness which we call hyperchaos. Indeed random-random walks in ordinary space, as diffusion in disordered systems, have shown a $1/f$ behaviour [1-3]. Thus hyperchaos here introduced is a random-random walk in phase space, where in fact one of the two sources of complex behaviour is due to the fractal structure arising from deterministic dynamics.

To evaluate the impact of this letter, we premise some historical remarks on $1/f$ spectra in nonlinear dynamics.

Some years ago it was discovered [4] that, in a nonlinear dynamical system with more than one attractor, introduction of random noise induces a hopping between different basins of attraction, giving rise to a low-frequency spectral divergence, resembling the $1/f$ noise well known in many areas of physics [5]. Such a discovery was confirmed by a laser experiment implying two coexisting attractors [6] and later the effect was observed in other areas as *e.g.* Josephson tunnel junctions [7, 8].

The effect was questioned with two objections:

a) a noise-induced jump across a boundary leads to a telegraph signal, hence to a single Lorentzian spectrum [9*a*];

(*) Present address: Laboratory for Computer Science, MIT, Cambridge, MA.

b) a computer experiment yielded a power law only over a limited spectral range [9*b*].

The questions were answered [9*c*] with a statement of the empirical conditions under which the $1/f$ spectra appeared, namely:

i) coexistence of at least two attractors (so called «generalized multistability» [6]);

ii) presence of noise;

iii) some «strangeness» in the attractors.

As a matter of fact this third condition was rather vague. To make it more precise, two theoretical models were explored, namely a one-dimensional cubic iteration map with noise [10] and a forced Duffing equation with noise [11]. Both these papers disclose interesting features, bringing more light on the above assumption iii). Figure 2 of ref. [10] shows that the size of the $1/f$ spectral region increases with the r.m.s. of the applied noise, that is with the probability of crossing the basin boundary by a noise-induced jump (fig. 3 of the same reference shows that the Liapunov exponent approaches the crisis value for increasing noise).

The numerical evaluation of ref. [11] showed that for some control parameters the boundary between basins of attraction was an intricated set of points, through which it was impossible to draw a simple line. In such cases the noise was most effective in yielding low-frequency spectra $1/f$-like.

On the other hand, a fundamental logical approach to the $1/f$ problem was based on the composition of a large number of Lorentzians (or elementary Markov processes with exponential decay) whose weights are log-normally distributed [12], thus fulfilling the relation

$$\int_{\gamma_1}^{\gamma_2} \frac{\gamma}{\omega^2+\gamma^2}\, p(\gamma)\,\mathrm{d}\gamma \simeq \mathrm{const}\, x \frac{1}{\omega}\,, \tag{1}$$

provided $p(\gamma) \sim 1/\gamma$, and for the frequency range $\gamma_1 \ll \omega \ll \gamma_2$.

Motivated by the rate process considerations on ref. [9*a*], which yielded a single Lorentzian for two attractors, we developed a kinetic model [10] based on a single transition rate for each pair of attractors. In the case of M attractors, this yielded $M-1$ Lorentzians. To approximate integral (1) by a sum (5% accuracy in fitting a $1/f$ law would require about one pole per decade) a large number $M \gg 2$ of attractors is necessary and hence the integral of eq. (1) would be replaced by the sum over the $M-1$ Lorentzians corresponding to the eigenvalues of the kinetic model, however, there is no reason to weigh the Lorentzians according to their reciprocal widths, hence no satisfactory reconstruction of a $1/f$ spectrum was possible. In fact, an experiment on a forced and noisy Duffing oscillator with an increasing number of attractors [13] did not offer a clear evidence of the expected scaling of the spectral exponent with the number of attractors. On the contrary, ref. [11] showed that the boundary region between just two attractors was suffucient to yield $1/f$-like spectra, at variance with the many-attractor model. Thus this suggested that the boundary structure was the real responsible for a large number of decay constants (possibly log-normally distributed).

In the meantime, the fractal structure of a basin boundary was explored in some examples [14, 15]. This means the following. As the phase point wanders within one basin of attraction, if we draw a sphere around the point defining its distance from the other basin of attraction, the radii of these spheres are distributed with all scale lengths, according to the self-similar structure of the fractal boundary. If we consider two-dimensional projections of the phase space as in fig. 3 to 6 of ref. [11], the spheres will be circles.

Based on the above considerations, we have built an elementary cellular automaton which models the motion of the phase point within a fractal basin boundary under the presence of random noise. We model the boundary region of two basins of attraction A and B as two adjacent one-dimensional lattices of sites. Suppose we start from site i. At each discrete time step, if i belongs to A $(i \equiv i_A)$, it moves one step forward on the same lattice $(i_A \to (i_A + 1)_A)$ and, if it belongs to B, it goes one step backward $(i_B \to (i_B - 1)_B)$. In the absence of noise, once the motion has started on one basin, it will remain on it forever. In the presence of noise, at each time step there is a finite probability of a «cross» jump at the same lattice site, from strip A to B: $i_A \to i_B$.

We call L the maximum size of the boundary region and $l_i \leq L$ any of the possible sizes of the fractal set. At each time step, the probabilities of permanence and jump are, respectively,

$$P_{AA} = P_{BB} = l_i/L \ , \qquad P_{AB} = P_{BA} = 1 - l_i/L \ . \tag{2}$$

To build a self-similar structure we allow l_{i_k}/L to scale as $l_{i_k}/L = (1/2)^{V(i_k)}$, where $V(i_k)$ is a natural number sorted randomly for each site i_k ($i = -\infty$ to ∞, $k = A, B$). To deal with a real numerical experiment, we consider finite sequences of N sites (*e.g.* $N = 10^3$) and we truncate the fractality by imposing $0 \leq V(i_k) < F$. Here F is a finite integer denoting the maximum partitioning $(1/2)^{F-1}$, that is the ultimate resolution of the measuring device in appreciating the fractal structure of our set. With all this in mind, for each evolution we extract a double sequence of N integers randomly distributed between 0 and $F-1$, and denote each site i_k by the corresponding number $V(i_k)$. This means that we have attributed to each site an «area of respect», that is a specific separation l_{i_k} from the other attractor, with l_{i_k} depending on $V(i_k)$ as shown above. We start, *e.g.*, on the basin A from $i_A = N/2$.

At each step, to account for noise, we generate a random number y uniformly distributed between 0 and 1. If $y \leq (1/2)^{V(i_A)}$, then at the next time the point goes to $i_A + 1$ on attractor A; if $y > (1/2)^{V(i_A)}$, then the point jumps instantaneously to site i_B and at the next time it goes to $i_B - 1$ on attractor B.

By measuring the position co-ordinate, taking the Fourier transform and squaring it, we can build the power spectra, that is the transforms of the position correlation functions.

In fig. 1 we show four power spectra for $F = 4$, 6, 10 and 14, respectively. In fact, we have measured spectra for all fractalities between $F = 4$ and $F = 14$, but we report four samples over slightly more than three frequency decades. The sequence shows that, as the fractality increases, the slope of the log-log plot goes from about 2 (single Lorentzian) to about 1 ($1/f$ spectrum). This appears better in fig. 2, where the slope α of the $f^{-\alpha}$ spectral law is plotted *vs.* the fractality F. The Lorentzian ($\alpha = 2$) of the random telegraph model [9*a*] is easily recovered for $F = 1$, thus showing that noise-induced jumps across a regular line boundary fulfill the intuitive expectation of a single decay rate. An analogy with the random-random walk [1-3] is easily drawn. Indeed our motion is bound with a r.m.s. deviation going from about $\sqrt{t}$ to $|\log t|^2$ as the fractality F increases from 4 to 14, according to Sinai law [1]. The corresponding data will appear in a longer report.

For comparison we mention other approaches leading to $1/f$ or anyway non-Lorentzian low-frequency spectra:

i) Pomeau-Manneville type-3 intermittency corresponds to slowly diverging trajectories with a $1/f$ power spectrum [16, 17]. This behaviour is intrinsic to the dynamics, hence it occurs without noise.

ii) A deterministic diffusion process may occur beyond «crisis» [18] when two

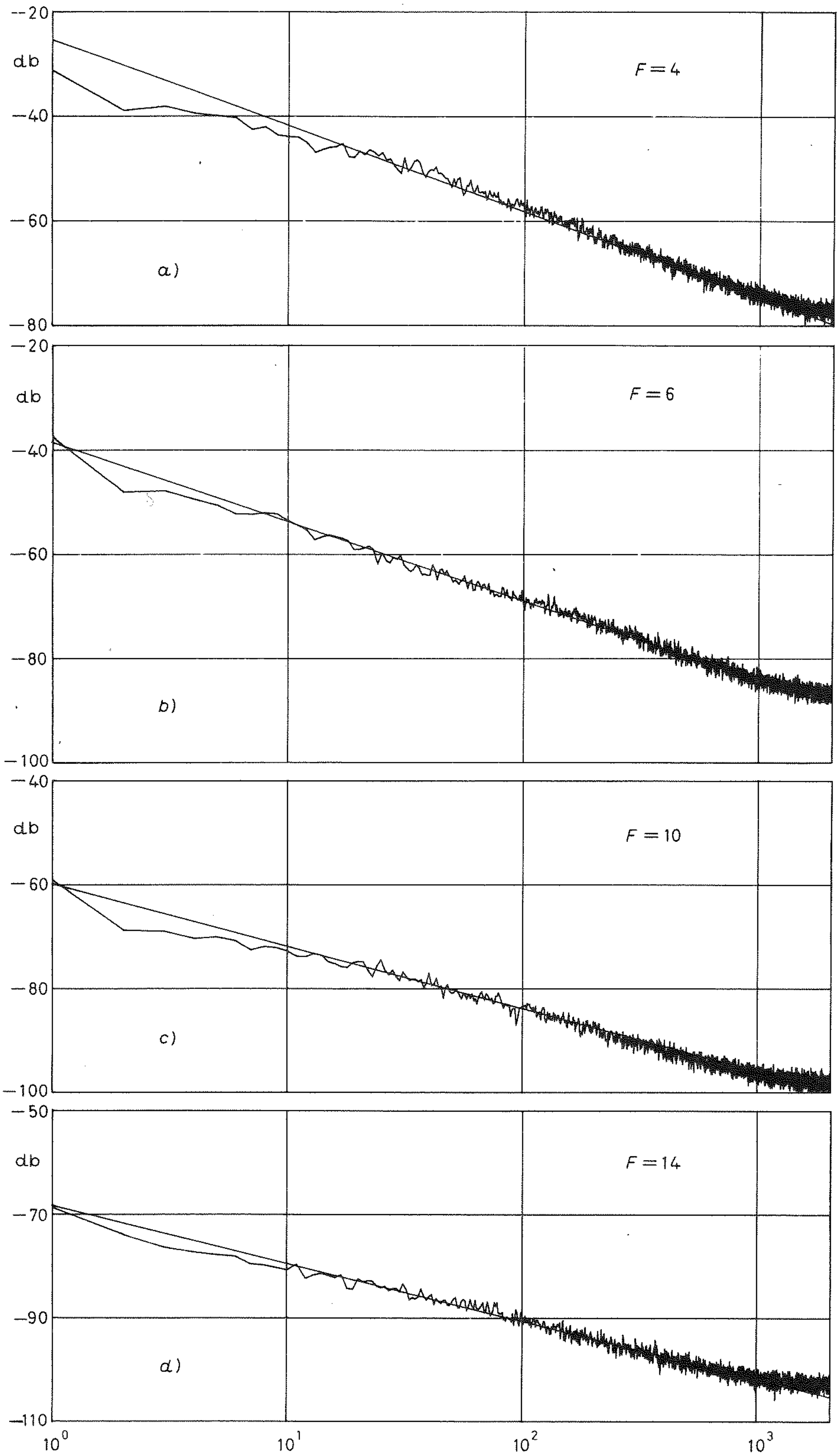

Fig. 1. – Power spectra (vertical) *vs.* frequency (horizontal) in log-log scale. Wavy lines: measured spectra; straight lines: best fits, whose slopes α are reported in the next figure. The four samples shown refer to $F=4$, 6, 10 and 14, respectively.

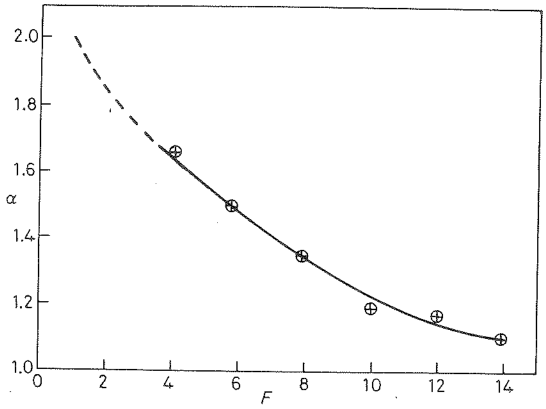

Fig. 2. – Exponents α of the power law $f^{-\alpha}$ *vs.* fractality F.

otherwise disjoint attractors merge into a single one. Here again no noise is required and a comparison of this behaviour with noise-induced jumps was given in ref. [11].

iii) Another comparison of intrinsic *vs.* noise-induced intermittency was carried on for a damped driven pendulum, which models a Josephson junction [19]. This last paper offers numerical evaluations of spectra, showing an $1/f$ region extending over two decades, but to our knowledge nobody has tried so far to analyse the role of fractality and draw a comparison with Sinai subdiffusive motion, as we did here.

Among other things, the results of this paper may strongly affect our current understanding of optical bistability (OB) phenomena, OB is described in terms of two fixed-point attractors, which, however, are the result of a collective dynamics implying many degrees of freedom. There are no exhaustive analyses of the structure of the basin boundary, thus possible fractal structures may appear if the dynamics is evaluated in detail. On the other hand, in order to reduce the signal power necessary to drive the OB device from one state to the other, the system is usually set very near to the boundary. Thus unavoidable random noise might induce low-frequency spectra of the type described above.

* * *

This work was partly supported by the EJOB Program of the European Economic Community. We are grateful to M. F. SHLESINGER and A. POLITI for useful discussions.

REFERENCES

[1] SINAI YA. G., in *Proceedings of Berlin Conference on Mathematical Problems in Theoretical Physics*, edited by R. S. SCHRADER *et al.* (Springer, Berlin) 1982, p. 12.
[2] MARINARI E., PARISI G., RUELLE D. and WINDEY P., *Phys. Rev. Lett.*, **50** (1983) 1223.
[3] NAUENBSERG M., *J. Stat. Phys.*, **41** (1985) 803.
[4] ARECCHI F. T. and LISI F., *Phys. Rev. Lett.*, **49** (1982) 94.
[5] DUTTA P. and HORN P. M., *Rev. Mod. Phys.*, **53** (1981) 497.
[6] ARECCHI F. T., MEUCCI R., PUCCIONI G. P. and TREDICCE J. R., *Phys. Rev. Lett.*, **49** (1982) 1217.
[7] MIRACKY R. F., CLARKE J. and KOCH R. H., *Phys. Rev. Lett.*, **50** (1983) 856.
[8] KAUTZ R. L., *J. Appl. Phys.*, **58** (1985) 424.