THE PHYSICS OF LASER CHAOS

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The practical problems arising in the experimental implementation of laser chaos are discussed here, with particular reference to the recent observation of Shilnikov chaos.

1. INTRODUCTION

The physics of deterministic chaos and related problems has been covered in many recent symposia and review contributions, both in its theoretical and experimental aspects. In particular, the rapidly growing interest for chaos in quantum optics is responsible for a large number of findings. The progress up to last year is summarized in a collective volume /1/. Here I present a short review of the activity of my research group, focussed on single mode, homogeneous line, lasers (Sect. 2). Then (Sect. 3) I describe our latest result, namely, a method of return time statistics which is the most efficient way of characterizing a Shilnikov instability. This instability, introduced as a theoretical object and observed qualitatively in chemical instabilities, is supported by fully quantitative evidence from recent observations on a laser system /2/ to which our statistical method is applied here.

2 CHAOS IN LASER SYSTEMS

A comprehensive review of this field is now available /1/. here I present a short account of what has been done in the Florence group.

2.1 Various classes of chaos

The similarity between Maxwell-Bloch (MB) equations for a single laser mode resonant and with a homogeneous gain line and Lorenz equations suggests the easy appearance of chaotic instabilities in single-mode, homogeneous-line lasers /3/. However, time scale considerations rule out the full dynamics for
most of the available lasers. Lorenz equations have damping rates within one
order of magnitude. On the other hand, in most lasers the three damping rates
are wildly different from one another.

Calling, as usual, Υ⊥, Υ∥ and k the three damping rates for polarization,
population and field, the following classification has been introduced /4/

Class A (e.g. He-Ne, Ar, Kr, dye): Υ⊥ ≈ Υ∥ >> k

The two last MB equations can be solved at equilibrium (adiabatic
elimination procedure) and one single nonlinear field equation
describes the laser. N=1 means a fixed point attractor, and hence
coherent emission.

Class B (e.g. ruby, Nd, CO₂): Υ⊥ >> k ≈ Υ∥

Only polarization is adiabatically eliminated and the dynamics is
ruled by two rate equations for field and population. N=2 allows also
for periodic oscillations.

Class C (e.g. FIR lasers) Υ∥ ≈ Υ⊥ ≈ k

The complete set of MB equations has to be used, hence Lorenz like
chaos is feasible.

We have carried out a series of experiments on the birth of deterministic
chaos in CO₂ lasers (Class B). In order to increase the number of degrees of
freedom by at least, we have tested the following configurations:

(i) Introduction of a time-dependent parameter to make the system non-
autonomous /5/. Precisely, an electro-optical modulator modulates the
cavity losses at a frequency near the proper oscillation frequency
provided by a linear stability analysis, which for a CO₂ laser happens to
lie in the 50-100kHz range, making an accurate set of measurements easy.

(ii) Injection of the signal from an external laser detuned with respect to
the main one, choosing the frequency difference near the above-mentioned
. With respect to the external reference the laser field has two
quadrature components which represent two dynamical variables. Hence we
reach N = 3 and observe chaos /4,6/.

(iii) Use of a bidirectional ring, rather than a Fabry-Perot cavity /7/. In
the latter case the boundary conditions constrain forward and backward
waves, by phase relations at the mirror, to act as a single standing
wave. In the former case, forward and backward waves have to fill the
total ring length with an integer number of wavelengths but there are no mutual phase constraints, hence they act as two separate variables. Furthermore, when the field frequency is detuned with respect to the center of the gain line, a complex population grating arises from interference of the two counter-propagating waves, and as a result the dynamics becomes rather complex, requiring $N = 3$ dimensions.

(iv) Addition of an overall feedback, besides that provided by the cavity mirrors, by modulating the losses with a signal provided by the output intensity /8/. If the feedback has a time constant comparable with the population decay time, it provides a third equation sufficient to yield chaos. Notice that while methods (i), (ii) and (iv) require an external device, (iii) provides intrinsic chaos. In any case, since feedback, injection or modulation are currently used in laser applications, the evidence of the existence of chaotic regions is a warning against optimistic trust in laser coherence.

Of course, the requirement of three coupled nonlinear equations does not necessarily restrict the attention to just Lorenz equations. In fact none of the explored cases i) to iv) corresponds to Lorenz chaos.

2.2 Role of noise in chaos: Hyperchaos

Nonlinear dissipative systems can have many simultaneously coexisting basins of attraction (generalized multistability /5/). This situation can be destabilized by changes of the control parameters, merging two independent attractors into a single one via an intermediate region which is only sporadically visited near the transition. The associated dynamics implies a low frequency tail (deterministic diffusion) /9/; conversely, when the above coexistence is stable, application of external noise may induce jumps between two otherwise disjoint regions of phase space.

The simultaneous presence of deterministic chaos and noise should not introduce new features within one attractor, since trajectories are already irregular. When however many attractors coexist for the same parameters, addition of noise makes it possible to leave a basin and go to another (which would be otherwise forbidden by the uniqueness theorem). This "hyperchaos" gives rise to low frequency spectra of $1/f$ type.
Clear evidence of generalized multistability was first shown in an electronic oscillator /10/ and then in the modulated laser /5/. In both cases, besides the qualitative appearance of different attractors in phase space, there was a low frequency spectral component due to noise-induced jumps between different attractors. Both measurements, however, might be considered as experimental artifacts. In fact, there is evidence of single attractors made of two sub-regions with infrequent passages from one to the other (see e.g. the Lorenz attractor). In such a case, the low-frequency tail corresponds to the sporadic passages, and does not require added noise (deterministic diffusion). As a matter of fact, power spectra do not permit discrimination between the two phenomena.

An analysis of the role of noise in GM was given for a cubic iteration map, allowing for two simultaneous attractors, and in the numerical studies of a forced Duffing oscillator with a double potential well, in a parameter region allowing for the simultaneous existence of more than one attractor /11/.

A recent solution of the 1/f spectral problem /12/ is based on the double randomness due to the irregular deterministic motion with long-lived transients, where the trajectory wanders near a fractal basin boundary, and in the presence of stochastic noise. In this case, 1/f behavior can be accurately traced over more than 3 decades in frequency.

3. LASER DYNAMICS WITH COMPETING INSTABILITIES

By operating a CO$_2$ laser with feedback in a parameter range with three coexisting fixed points, we have found experimental evidence of competition between different instabilities, displaying the successive transition from a Hopf bifurcation to Shilnikov chaos and eventually to regular spiking, as a control parameter is monotonically increased. Each of these phenomena is due to the dominant attraction of one of the three fixed points. A global description could be achieved by composition of successive maps, each one describing the re-injection from and to a box around each of the fixed points /13/. However, because of the dominant character of one of the instabilities in each different range of the control parameter, we adopt a piecewise interpretation, in which we attribute most of the return time in a suitable
Poincaré section to only one fixed point.

As stated in the previous section, a single-mode class B laser is governed by two coupled degrees of freedom (intensity $x$ and population inversion $y$). Introduction of a third degree of freedom $z$ by feedback leads to a three-dimensional system displaying oscillatory instabilities and chaos /8/.

The dynamical equations are given in /8/, where we also discuss the details of the feedback scheme. There are three control parameters, the bias $B$, and the gain $r$ and damping rate $\beta$ of the feedback loop, all the other parameters being fixed at the values reported in /8/.

The experimental information consists of phase space projections on the $x$-$z$ plane, obtained by feeding an $x$-$y$ oscilloscope with a photodetector signal proportional to the laser intensity and with the feedback signal applied to the modulator. For each phase space portrait the associated time plot $x(t)$ is recorded on a digital oscilloscope.

In Figs. 1 and 2 we report experimental data for two different gain values, and in each case for three increasing $B$ values. In the phase space portraits as well as in the time plots, 0, 1 and 2 denote the three fixed points. In Fig. 1 (low gain) as $B$ increases, the first fixed point 1 becomes unstable through a Hopf bifurcation (Fig. 1b). For higher gain, the limit cycle becomes unstable and gives rise to chaotic trajectories, as we will see below in discussing Fig. 2b. As the loop gets wider, it eventually approaches 0 (the zero intensity solution). This implies a long pause, since the laser is nearly extinguished and must wait for a long build-up time before revisiting the region of phase space where the other fixed points are located. This pause has a stabilizing action since it washes out any memory of fluctuations associated with the previous cycle. Eventually the return time is dominated by this pause. The loop then corresponds to narrow spikes, slightly perturbed by the attraction of point 1, plus a long lethargy time in the vicinity of point 0 (Fig. 1c).

For high gain, two new important features appear. First (Fig. 2a) the Hopf bifurcation around 1 gives rise to a subharmonic route to a local chaos. As the loop widens, it is attracted by the fixed point 2 (Fig. 2b). The trajectories around 2 are subjected to large temporal fluctuations while spiralling (Shilnikov chaos). As the phase point approaches 2, the escape
FIGURE 1
Laser with feedback. Phase space projections z-x (feedback voltage - laser intensity) and time plots of the intensity x(t) for low feedback gain. Intensity increases downward. Normalization for B as in /21/; a) B=0.259; b) B=0.274; c) B=0.385. Approximate locations in the phase plane of points 1 and 0 are indicated.
FIGURE 2
Phase space projections z-x (feedback voltage - laser intensity) and time plots of the intensity x(t) for high feedback gain. a) B=0.296; b) B=0.311; c) B=0.411. Approximate locations of points 0, 1 and 2 in the phase plane are indicated.
time gets longer, so that eventually the time spent around 2 is longer than that around 1 or around 0, thus characterizing the global behavior (Fig. 3). But, once 0 has been reached, its lethargy cancels the memory of the fluctuations around the Shilnikov instability, thus regularizing the return time and giving rise to narrow and equally spaced spikes (Fig. 2c).

If we now disregard the local features of the phase space motion and look for a global indicator, the most convenient one is the return time after one whole loop. This is measured experimentally by an average rate meter. Figure 4 shows the behavior of the average Poincaré frequency \( f_p \) (reciprocal of the return time) versus the bias \( B \), for several gain values.

For low gain, \( f_p \) has a monotonic decrease versus \( B \). Initially it corresponds to the Hopf frequency around 1, but, as the attraction of 0 prevails, \( f_p \) becomes the reciprocal of the occurrence time of the narrow spikes. For high gain, besides the irregular regime where \( f \) is undefined because of strong chaotic changes, a novel feature is the appearance of vertical step-like jumps, including hysteresis at each jump (see expanded inset in Fig. 4). This is equivalent to forbidden frequency values, just the opposite of what occurs in locking phenomena where there are horizontal steps in frequency.

In summary, the global features of the phase space motion with competing instabilities can be divided into three regimes, each corresponding to a dominant fixed point:

I) For low \( B \) the motion starts at 1 and for increasing \( B \) we have a Hopf bifurcation, followed by a subharmonic route to chaos.

II) For high \( B \), the motion is a regular periodic spiking with no memory of other features (0 dominant).

III) At intermediate \( B \) (2 dominant), a Shilnikov instability provides chaotic fluctuations in the return time.

We can treat regime II) in the limit of large \( B \), where the feedback voltage \( z \) can be adiabatically eliminated. In order to cope with the sharp spikes, it is more convenient to switch notation to a log representation of the intensity /14,15/. The two remaining equations are
\[ \frac{\dot{x}}{k} = -x(1 + \alpha \sin^2 z) + xy, \]  
\[ \frac{\dot{y}}{\gamma_r} = -x + A - xy, \]  
where the feedback voltage \( z \) is 
\[ z = B - rx. \]  
Now, with the transformation 
\[ s = \ln x, \]  
and in the absence of feedback \( (\alpha = 0) \) Eqs. (1) reduce to 
\[ \frac{d^2s}{d\tau^2} = -\epsilon \frac{ds}{d\tau} (1 + e^s) + A - 1 - e^s, \]  
where \( \tau = \sqrt{k\gamma_r} t \) and \( \epsilon = \gamma_r/k \) are scaled parameters. 

For a small ratio \( \epsilon \) between population and photon decay rates (this ratio is \( 10^{-3} \) for our CO$_2$ laser) the equation reduces to that of a lossless Toda oscillator whose period is easily evaluated. It is in fact given by /15/ 
\[ T = \int_0^s \frac{ds}{(E + (A-1)s - e^s)^{1/2}}, \]  
where the constants \( E \) and \( s_0 \) are adjusted to experimental situations. By extending the above treatment to include the feedback we obtain an increasing Poincaré period \( T' \) which goes approximately as 
\[ \frac{T'}{T} \propto \frac{1}{(1 - \alpha \sin^2 B)^{1/2}}, \]  
in qualitative agreement with the experimental data of Fig. 4, curve a.

In regime III we take the opposite limit, namely, that most of the return time is spent within a small distance of fixed point 2. The time \( \tau \) spent in the unit box around that point is shown in /13/ to vary as \( \tau \propto \ln(1/z_0) \) where \( z_0 \) is the offset at the box entrance along the expanding direction. As the system moves from 1 dominant to 2 dominant, it is reasonable to take \( 1/z_0 \) proportional to the bias \( B \), since for larger \( B \) the
Phase space projection and intensity vs. time x(t) for the same gain as Fig. 2 and for B=0.351. The time expanded plot x(t) shows clearly the role of the Shilnikov instability in yielding chaotic return times. Approximate locations of points 0, 1 and 2 in the phase plane are indicated.

Plot of the Poincaré frequency $f_p$ versus bias B for four different gain values. Curves a and d refer to the gain values of Fig. 1 and Figs. 2,3, respectively. Curve c displays a step-like feature with hysteresis, as clearly shown in the expanded inset of the square region. The shaded region refers to a chaotic $f_p$, hence the traces reported within that region are just a single scan.
phase point comes closer to fixed point 2. Hence, the above solution yields a
frequency \( f_p = 1/\tau \) monotonically decreasing with \( B \), in qualitative agreement
with the average trend of Fig. 4. However, to explain the steplike details,
we must consider two contracting directions, that is, the three-dimensional
character of the spiral at the Shilnikov instability. In this case the
corresponding return map /13/ can be modeled in terms of a number of loops
corresponding to the number of jumps in the expanded plot of Fig. 4, thus
making it possible to evaluate the complex contraction rate and the real
expansion rate of the Shilnikov instability.

Finally, let me stress the conceptual difference between Hopf chaos
(Fig. 2a) and Shilnikov chaos (Fig. 3). The former case is the end point of a
subharmonic sequence. Many amplitudes appear, corresponding to different
loops around the unstable point 1.

The latter case yields practically equal-height pulses but with
different time separations. For Shilnikov chaos, the interplay between the
interaction along the stable manifold and the expansion along the unstable
direction manifests itself in a spread of return times at the same Poincaré
section. Such a situation is the natural evolution of the staircase region of
Fig. 4c,d, for higher gain values. In such a case, the statistical properties
of the return time allow reconstruction of the theoretical map of /13/.

A detailed correspondence between the model of /13/ and our experimental
time statistics is given in a forthcoming paper /16/, together with a global
theoretical picture of the phenomena reported here.

REFERENCES

1) F.T. Arecchi and R.G. Harrison (Eds.), Instabilities and Chaos in Quantum
2) F.T. Arecchi, R. Meucci and W. Gadomski (to be published).