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Experimental Characterization of Shil'nikov Chaos by Statistics of Return Times.

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Abstract. – We report experimental evidence of a new type of chaos characterized by pulses equal in shape, but irregularly separated in time. The times of return to a Poincaré section are statistically spread, however their iteration map is one-dimensional and in close agreement with that arising from Shil'nikov theory. Thus, the iteration map of the time intervals becomes the most appropriate indicator of this chaos. The residual width of the experimentally measured maps is due to a transient fluctuation enhancement peculiar to macroscopic systems, which is absent in low-dimensional chaotic dynamics.

We report experimental evidence of quasi-homoclinic behaviour characterized by pulses with regular shapes but chaotic in the distribution of their times of occurrence. The regularity of shape means that the points at any Poincaré section are so closely packed that extremely precise measurements of their position would be required to differentiate one pulse from another. Instead, return times to a Poincaré section close to the unstable point display a large spread, due to the sensitive dependence of the motion upon the intersection coordinate. We, therefore, introduce that spread as the most suitable indicator of this chaos. Our experimental data yield iteration maps of return times in close agreement with those arising from the theory of Shil'nikov chaos [1, 2]. However, the theory must be supplemented by the consideration of experimental iteration maps of finite thickening independent of the accuracy of measurement. This is due to a transient fluctuation enhancement discovered earlier in the decay of an unstable state of a macroscopic system [3]. The latter phenomenon unavoidably introduces further statistical fluctuations into the chaotic dynamics of a macroscopic system. Large fluctuations of this type were first observed in the switch-on of a laser [3*a*] and then in many quenching phenomena such as spinodal decomposition and superfluorescence [3*b*].

Our experimental evidence of a Shil'nikov-type instability is based on a quantum optical system, namely a laser with an overall feedback. Precisely, we work on a single-mode CO_2

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laser with an intracavity electro-optic modulator yielding cavity losses proportional to the laser output intensity [4]. In an appropriate range of the control parameters, the dynamical behaviour consists of closed orbits in a three-dimensional phase space, visiting successively the neighbourhoods of three unstable stationary points, one of which is a saddle focus (a stable focus with an expanding direction perpendicular to it). The competition of the three instabilities in controlling the global features of the motion was described elsewhere [5]. Here we adjust the control parameters in order to have a dominance of the saddle focus, so that the motion consists of a quasi-homoclinic orbit asymptotic to it. Calling $-|\alpha|$ the real part of the complex conjugate eigenvalues and γ the positive eigenvalue corresponding to the unstable direction, Shil'nikov [1] showed that for $|\alpha| < \gamma$ there exists a countable set of unstable trajectories close to the homoclinic one. This structure of the flow is one of the simplest capable of generating chaotic behaviour in many autonomous systems like for instance the Lorenz model [6] or the Belousov-Zhabotinskii reaction [7].

In fig. 1 we report experimental plots of the laser intensity *vs.* time for two slightly different conditions. Keeping a fixed population inversion (pump 1.5 times above the threshold value) the system has two control parameters, the bias B and the gain r of the

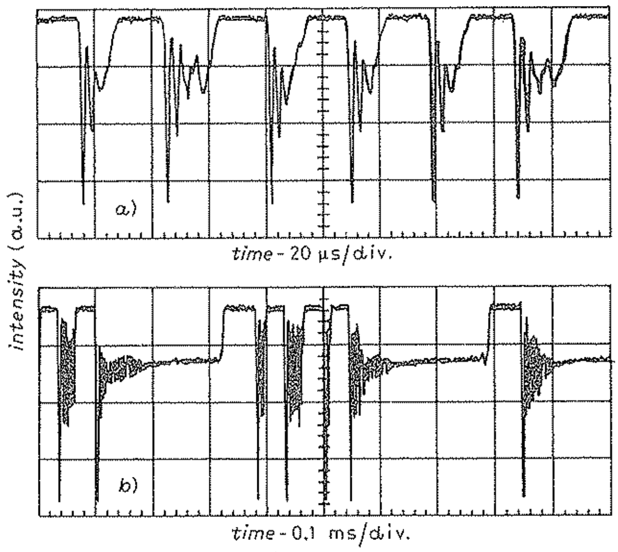

Fig. 1. – Time plots of the laser intensity in the regime of Shil'nikov chaos. *a*) and *b*) refer to the same B value ($B = 0.427$), but two different gains r (0.487 and 0.696, respectively) of the feedback loop. *b*) shows two long transients corresponding to a large number of small spirals around the saddle focus.

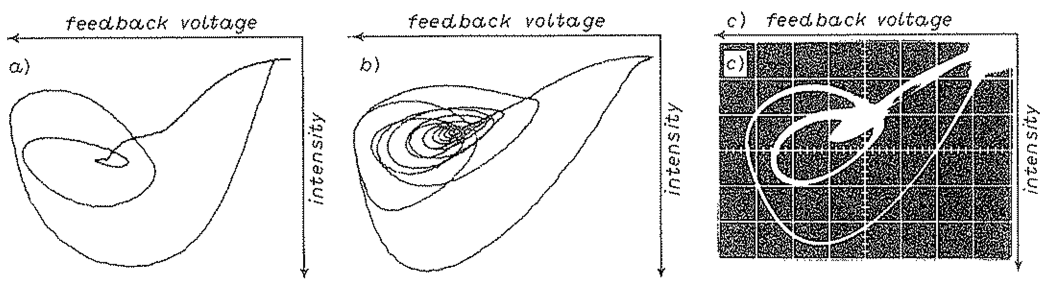

Fig. 2. – Phase space projections x-z (laser intensity-feedback voltage). *a*) and *b*) are single orbits obtained by a digitizer, referring to the same parameters of fig. 1*a*) and *b*), respectively. *c*) is the superposition of 30 000 orbits of type *a*).

amplifier driving the electro-optic modulator. We have kept B fixed (1) at $B = 0.380$ and increased r from $r = 0.467$ in fig. 1*a*) to $r = 0.491$ in fig. 1*b*). Figure 1*b*) shows clear evidence of a homoclinic orbit in the two long transients, which provide a lengthy permanence in a phase space region of almost constant intensity. This appears more clearly in the corresponding phase space projections (fig. 2*a*) and *b*)). For comparison we give in fig. 2*c*) a photographic exposure (over 1 s) of 30 000 orbits as that of fig. 2*a*), to show the stability of shape.

We see that the first three large oscillations of fig. 1 have strong anharmonic distortions and display common features over different repetitions, while the small oscillations around the saddle focus display slight differences from pulse to pulse. These latter oscillations are ruled by the linearized dynamics which consists [2] of a contracting spiral $\exp[-|\alpha|t]\cos(\omega t)$ on the stable manifold and of an expansion $\exp[\gamma t]$ along the unstable direction.

If we build a small cubic box of unit side centred at the saddle focus and oriented along the eigenvectors ξ, η, ζ, any tiny difference in the entrance coordinate along the expanding axis ζ will strongly influence the residence time inside the box and hence the spacing from the next re-injection.

We measure the time spacings by setting a threshold circuit near the top of the largest peak of the intensity signal. A time-to-amplitude converter (TAC) yields the sequence τ_i of successive time spacings, which is then classified as a statistical distribution by a multichannel pulse height analyser, or stored in a digitizer, so that correlation functions or iteration maps can be sorted out.

The statistical distribution of return times is a broad featureless curve which does not offer cues on the ordering of τ_i. On the contrary, the iteration map (τ_{i+1} *vs.* τ_i) displays a regular structure (fig. 3*a*)). To check whether we are in the presence of a one-dimensional (1D) iteration map, and the remaining thickness is due to the observation technique, or the

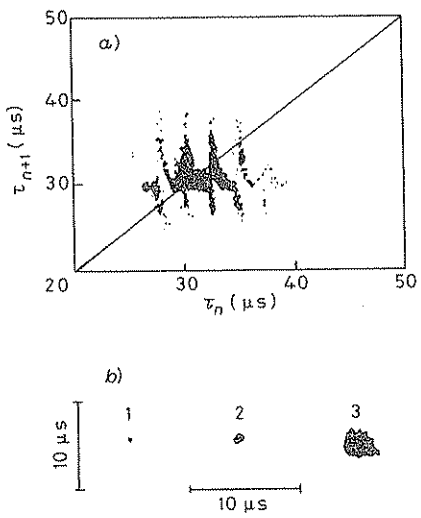

Fig. 3. – Experimental iteration maps of the return times. *a*) refers to $r = 0.487$ and to $B = 0.350$. *b*) shows the maps corresponding to regular periodic situations, namely, 1) an electronic oscillator, 2) the laser in a regular periodic regime and 3) the laser just at the onset of the instability but still with a regular period.

(1) The conversion factors from the laboratory parameters to the numbers given in the text for B and r have been derived in ref. [4] (there r was called f).

map is more than 1D, we report in fig. 3*b*) the iteration maps corresponding to three regular situations. In the absence of fluctuations in τ_i they should be pointlike (the image of a stable fixed point). In fact 1) corresponds to an electronic oscillator and it just shows the resolution of the TAC, 2) corresponds to the laser in a regular periodic regime away from the Shil'nikov instability, and 3) corresponds to the laser on the verge of the instability but still with a regular period. In this last case, the fluctuation associated with the nearby transition shows that, even without chaos in the return time, the close approach to an instability point introduces a fluctuation enhancement, which has no theoretical counterpart in the current treatment of deterministic chaos. To deal with this broadening, the dynamical equations should include a statistical spread in the injection coordinate at the Poincaré section near the saddle focus, to account for the macroscopic character of the experimental system. As it was shown in ref. [3], even though this spread has no relevance on the average dynamics, it contributes a large transient fluctuation whenever the system decays from an unstable point.

From a theoretical point of view, a homoclinic orbit asymptotic to a saddle focus has been modelled in terms of the following 1D map [2]:

$$\zeta_{n+1} = \zeta_n^{\alpha/\gamma} \cdot \cos\left[\omega/\gamma \cdot \ln(\zeta_n)\right] + \varepsilon\,, \tag{1}$$

where we recall that γ and $-\alpha \pm i \cdot \omega$ are the eigenvalues of the linearized flow at the saddle focus, ζ is the coordinate along the unstable manifold and ε is the deviation along ζ from the homoclinic orbit at the Poincaré section in the neighbourhood of the saddle point ($\varepsilon = 0$ corresponds to the homoclinic condition).

The meaning of eq. (1) stems from the following considerations. As said above, the laser dynamics with three degrees of freedom corresponds to a trajectory in a 3D space. As shown in fig. 1 and 2, the time fluctuations are mainly due to one of the fixed points, namely the «spiral in» or saddle focus. Around this point, the stable directions lie on a plane ξ, η, so that the projected trajectory shrinks as $\exp[-\alpha \pm i\omega t]$ and the unstable direction lies on the perpendicular direction ζ, along which the trajectory diverges as $\exp[\gamma t]$. Hence, any deviation ζ_n from the stable plane is mapped after one orbit into a new value ζ_{n+1} given by eq. (1). Even if initially $\zeta(0) = 0$, the offset ε allows for a finite ζ after an iteration. The parameter ε is controlled via the bias B of the feedback amplifier.

As shown in fig. 2 it is experimentally difficult to resolve changes in ζ, while the orbital period τ is subjected to large fluctuation. Hence we use eq. (1) to derive an iteration map for τ. Since most of the time fluctuations are due to saddle focus, we can neglect the perturbations in τ due to the other fixed points, and relate the return coordinate ζ to the initial coordinate ζ_0 via the unstable eigenvalue γ as $\zeta = \zeta_0 \exp[\gamma\tau]$. Using this relation, we transform eq. (1) into an iteration map for the orbital periods (²):

$$\tau_{n+1} = -\ln\left[\exp[-\alpha/\gamma \cdot \tau_n] \cdot \cos[\omega/\gamma \cdot \tau_n] + \varepsilon\right] = -\ln(\varphi(\tau) + \varepsilon)\,, \tag{2}$$

where α, γ, ω and ε are the same as above, and we have collected in φ the terms which contain τ_n.

Comparison of eqs. (1) and (2) shows the enhanced sensitivity to fluctuations in ε of the τ map with respect to the ζ map. The sensitivities of the two maps are given, respectively, by $\partial\zeta/\partial\varepsilon = 1$ and

$$\partial\tau/\partial\varepsilon = [\varphi(\tau) + \varepsilon]^{-1}. \tag{3}$$

(²) Equation (2) would yield an oversimplified map. In fact, fig. 2 shows that the orbits undergo a strong anharmonic distortion due to the attracting role of the other points in phase space. To account for this distortion, we should add higher harmonics to the cosine term of eq. (2).

This sensitivity factor acts as a lever arm whenever $\varphi(\tau)+\varepsilon$ becomes very small. Notice that: i) This is not deterministic chaos; in fact, large fluctuations can be expected even for a regular dynamics implying a fixed point τ^* as shown in fig. 3*b*). ii) It is not associated with the homoclinicity condition $\varepsilon=0$; in fact, for finite ε there may be a τ^* such that $\varphi(\tau^*)+\varepsilon=0$.

Since a homoclinic orbit is the dynamic counterpart of repeated decays out of an unstable state, the result is like re-positioning the initial condition in an experiment on a single decay. As a consequence, superposed to the deterministic dynamics (either regular or chaotic), the high sensitivity (3) may provide a broadening of the τ maps not detectable in the ζ maps, whenever noise in the offset ε is present.

In fact, the model description $\dot{x}=F(x)$ of a large system in terms of a low-dimensional dynamical variable x is just an ensemble averaged description, and residual fluctuations on position x must be considered at some initial time, even though the successive evolution is accounted for by a deterministic law. In our case such a fluctuation is a stochastic spread $\delta\varepsilon$ on the offset ε of the position ζ.

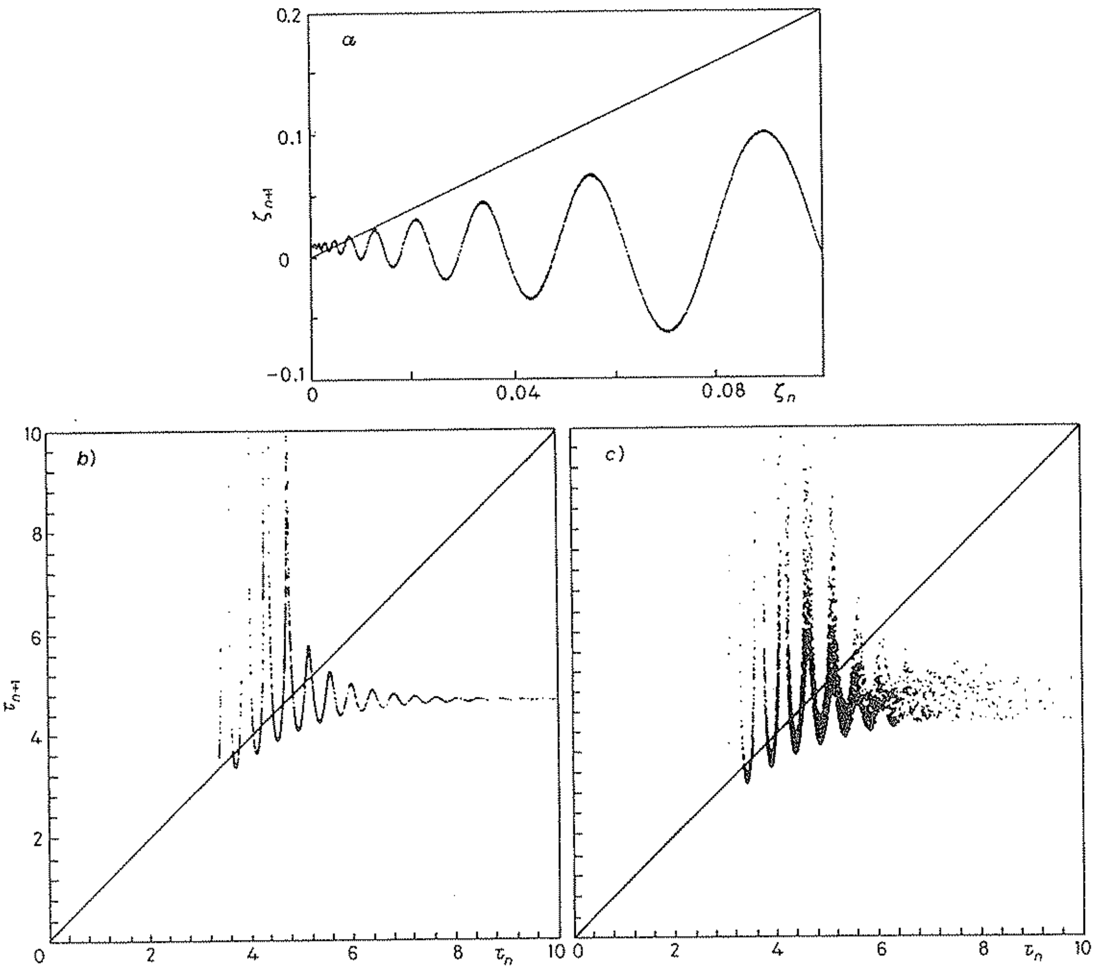

Fig. 4. – Numerical iteration maps for Shil'nikov chaos. Parameter values: $\omega/\gamma=13.0$, $\alpha/\gamma=0.986$, $\varepsilon=0.01$. *a*) is the ζ map in the presence of noise $\delta\varepsilon=10^{-2}$ (the corresponding map with zero noise is practically the same). *b*) and *c*) are the τ maps without and with noise $\delta\varepsilon=10^{-2}$, respectively.

As shown in fig. 4 the same amount of $\delta\varepsilon$ in eqs. (1) and (2) leaves unaltered the ζ maps, while it strongly affects the τ maps, making them appear like the experimental data.

If we specialize the map parameters to a regular orbit (fixed points both in ζ and τ spaces), introduction of $\delta\varepsilon$ does not broaden the ζ point, while the τ point broadens, in agreement with the experiment of fig. 3*b*). For example, the values $\alpha/\gamma = 0.98$, $\omega/\gamma = 2.98$ and $\varepsilon = 0.01$ yield one fixed point $\tau^* = 5.327$, with a sensitivity $\partial\tau^*/\partial\varepsilon = 182$.

Notice that the noise effect here reported has nothing to do with additive noise effects on return maps already described [8]. Indeed these latter ones refer to the scaling behaviour near stationary bifurcations, whereas our data refer to transient fluctuation enhancement and they do not leave a permanent mark (such as an orbital shift or broadening).

Thus, while Shil'nikov chaos is a deterministic effect described «on the average» by the backbone of the τ or ζ maps, the superposed thickening is a noise effect peculiar of τ maps and undetectable in ζ maps. This new effect is a specific indicator of intrinsic fluctuations, and it allows to draw a demarcation line between a real life experiment and a model simulation, where this second feature is absent.

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