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Statistical dynamics of class-B lasers

P. Paoli[1], A. Politi[1], and F.T. Arecchi[1,2]
[1] Istituto Nazionale di Ottica, Firenze, Italy
[2] Dipartimento di Fisica, Universitá di Firenze, Italy

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We present a simplified theory of the statistical behavior of a single mode laser ruled by two slow variables (class B lasers). By applying the center manifold theorem we introduce an improved adiabatic elimination procedure which reduces the description of class B lasers to two modified rate equations in an appropriate state space. The statistical dynamics is further reduced to a one dimensional Fokker-Planck equation in a parameter range corresponding to that where most experiments have been performed. The usefulness of this approach stems from the fact that a previously available theory is based on a two dimensional Fokker-Planck equation and limited to a narrower parameter range around threshold.

1. Introduction

Current semiclassical statistical theories of single mode lasers stem from the seminal paper by Risken [1], where the dynamics is accounted for by a 1-d Fokker-Planck equation for the intensity of the laser field. This is the only relevant variable because the coupled variables of the gain medium (polarization and population inversion) are taken to decay within sufficiently short time, so that their overall effect consists in introducing suitable nonlinear terms in the intensity equation. The lasers ruled by such a time scale, have been called class A [2] to distinguish them from those (class B) where the population damping is slow compared to the field evolution in the cavity, so that the laser dynamics is described by two coupled equations with comparable decay rates. As a consequence the statistical analysis is far more difficult than that for class-A systems in so far as the correspondent Fokker-Planck equation would be two-dimensional. A first systematic attempt of describing the behavior of class B lasers has been done by Morsch, Risken and Vollmer [3] who restricted their investigations to a small parameter region around laser threshold.

Here, we present a somehow complementary approach, well suited to describe laser dynamics at higher pump-values, where the deterministic evolution is equivalent to that of a weakly damped Toda oscillator [4]. In fact, the range of validity of the approach developed in this paper does not include either very small pump values (where, instead, [3] applies), or very large ones. In these two extremal conditions, dissipation plays a dominant role. More precisely, starting from the complete set of Maxwell-Bloch equations including stochastic sources, we perform a refined adiabatic elimination [5] which yields a set of modified rate equations. The resulting model is shown to belong to the well known class of weakly damped stochastic oscillators with, however, a crucial difference: the noise source, instead of acting directly on the momentum, acts on the 'position' of the oscillator. Accordingly, the standard procedure, based on the reduction to a 1-d Fokker-Planck Eq. [6] for the 'energy', has to be slightly modified. In Sect. 2, we develop the theoretical analysis, which leads to a perturbative expansion for the coefficients of the 1-d Fokker-Planck equation, and to the associated stationary distribution for the field intensity. In Sect. 3, a comparison with numerical simulations is done, which shows good agreement over a large range of parameter values.

2. General theory

The analysis of a single mode laser with a homogeneously broadened gain line is described by Maxwell-

Bloch equations for atomic polarization P, population inversion Δ, and electric field E [2]. When the cavity frequency is tuned at resonance with the gain line the equations reduce to

$$\dot{P} = -\gamma_\perp (P - E\Delta) + \frac{2g^2}{k}\sqrt{\frac{N_2}{D\gamma_\parallel}}\,\Gamma_P$$
$$\dot{\Delta} = -\gamma_\parallel (\Delta - D - 1 + DEP) + \frac{g^2}{\gamma_\perp k}\sqrt{\gamma_\parallel N}\,\Gamma_\Delta$$
$$\dot{E} = -k(E - P) \tag{2.1}$$

where noise sources for P and Δ have been added [3]; black body contributions to the field equation have been considered negligible. In the above equations, $\gamma_\perp, \gamma_\parallel$, and k are the decay rates of P, Δ and E respectively; g is the coupling constant between field and atoms; D is the pump parameter (i.e. $D+1$ represents the equilibrium value of Δ in the non-lasing state, and $D=0$ corresponds to threshold); finally, P and E have been normalized to their equilibrium values. The stochastic terms Γ_P, Γ_Δ have zero average and are δ-correlated

$$\langle \Gamma_P(t)\rangle = \langle \Gamma_\Delta(t)\rangle = 0$$
$$\langle \Gamma_P(t)\,\Gamma_P(t')\rangle = \langle \Gamma_\Delta(t)\,\Gamma_\Delta(t')\rangle =$$
$$= 2\delta(t-t'), \qquad \langle \Gamma_P(t)\,\Gamma_\Delta(t')\rangle = 0. \tag{2.2}$$

The coefficients in front of them are expressed in terms of decay rates, population N_2 of the upper atomic level, and total atomic population N. The strength of the stochastic sources have been derived in [7]. It is well known that, whenever $\gamma_\perp \gg \gamma_\parallel, k$, the polarization can be adiabatically eliminated, and the system (2.1) reduces to a set of two rate equations describing the energy balance between field and population inversion. Such a procedure is commonly used in the case of CO_2 lasers, although it is not completely correct, since $\gamma_\perp/k \simeq 10$. However, as long as $\gamma_\parallel \ll \gamma_\perp, k$, an alternative adiabatic elimination, based on the center manifold theory [8], allows to reduce the dimensionality of the problem yielding a set of modified rate equations. Thus far such a program was carried on mainly in the case of deterministic Eq. [9]. In this section we extend the method to the entire set of Langevin equations. According to [5], we perform a change of variables, such that the new axes are aligned along the eigenvectors of the stable fixed point $E_0 = P_0 = \Delta_0 = 1$,

$$R \equiv \frac{P - E}{1+\tilde{\gamma}} \tag{2.3a}$$
$$z \equiv \frac{P + \tilde{\gamma}E}{1+\tilde{\gamma}} \tag{2.3b}$$
$$w \equiv \frac{\Delta - 1}{\mu D} \tag{2.3c}$$

where $\tilde{\gamma} \equiv \gamma_\perp/k$, and

$$\mu = \sqrt{\gamma_\parallel(\gamma_\perp + k)/2D\gamma_\perp k} \tag{2.4}$$

represents the relevant smallness parameter, controlling the accuracy of adiabatic elimination. As a consequence, system (2.1) is rewritten as

$$\dot{R} = -\frac{(1+\tilde{\gamma})^2}{2\tilde{\gamma}\mu D}R + \frac{w}{2}(z-R) + \frac{\sqrt{Q\mu}}{2}\Gamma$$
$$\dot{z} = \frac{w}{2}(z-R) + \frac{\sqrt{Q\mu}}{2}\Gamma$$
$$\dot{w} = 1 - \mu w - (z-R)(z+\tilde{\gamma}R) + \sqrt{\frac{2\mu^3 D N_2 Q}{N}}\,\Gamma_w \tag{2.5}$$

where time has been rescaled as

$$\tau \equiv \sqrt{\frac{2D\gamma_\parallel \gamma_\perp k}{k+\gamma_\perp}}\,t \tag{2.6a}$$

and

$$Q \equiv \frac{16 g^4 N_2}{(k+\gamma_\perp)^2 \gamma_\parallel^2 D}. \tag{2.6b}$$

Γ, Γ_w satisfy (2.2), upon replacing t with τ. From (2.5) it is clear the existence of two well separated time scales. Indeed, for $\mu \ll 1$, the variable R exhibits a fast decay (time scale $O(\mu)$ compared with the motion of z, w (time scale $O(1)$). Therefore (neglecting the noise terms), it is reasonable to set the R derivative equal to zero. Such a procedure is approximately equivalent to applying the center manifold technique, which allows to find a perturbative expansion of the hypersurface where the asmptotic motion settles. Considering only the first order terms in μ, we obtain

$$R = \frac{\mu D\tilde{\gamma} w z}{(1+\tilde{\gamma})^2}. \tag{2.7}$$

In [5], other corrections to the final model have been estimated, which become relevant for $\tilde{\gamma} \leq 1$ yielding the second laser threshold. We now extend the adiabatic elimination in order to account for the stochastic forces. Limiting ourselves again to first order contributions, we can neglect the noise term on the first of (2.5), as it contributes only to the R-distribution, transversally to the centre manifold. Therefore, the final model is simply obtained by adding the effect of noise terms in the z, w-equations to the deterministic equations. By returning to the more familiar variable E, instead of z, we can write

$$\dot{E}=\frac{wE}{2}-\mu\frac{\tilde{\gamma}D}{(1+\tilde{\gamma})^2}E\left(1-E^2+\frac{w^2}{2}\right)+\frac{\sqrt{\mu Q}}{2}\Gamma$$
$$\dot{w}=1-E^2-\mu w\left(1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}E^2\right). \tag{2.8}$$

Moreover, since the expression of the center manifold is only approximated up to a first order in μ, the stochastic contribution to w has been neglected. In order to investigate the behavior of (2.8), it is convenient to introduce the new variable

$$q=\ln|E|. \tag{2.9}$$

Equation (2.8) then becomes

$$\dot{q}=w-\frac{\mu\tilde{\gamma}D}{(1+\tilde{\gamma})^2}\left(1-e^q+\frac{w^2}{2}\right)+\frac{\mu Q}{2}e^{-q}+\sqrt{\mu Q}\,e^{-q/2}\Gamma$$
$$\dot{w}=1-e^q-\mu w\left(1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}e^q\right). \tag{2.10}$$

In the r.h.s. of the q-equation we have added a deterministic contribution (directly proportional to Q) to account for an average effect of noise. It derives from the complex character of the electric field amplitude E [3], so far completely reglected. By setting $\mu=0$, we obtain the equations of a dissipation-less oscillator in a Toda-like potential. Therefore, system (2.10) belongs to the well known class of weakly damped stochastic nonlinear oscillators. It shows a relevant difference with the models usually studied in the literature [6], namely, noise acts on the 'position' q, rather than on the 'momentum' w, and hence, it can not be interpreted as a stochastic force due to the coupling with a thermal bath. However, in analogy with such models, it is possible to reduce the solution of (2.10) to that of a one-dimensional stochastic process. The main idea is to introduce again a slow variable, which, in the present case, turns out to be the pseudo-energy

$$W\equiv\frac{w^2}{2}+e^q-q-1\equiv\frac{w^2}{2}+V(q) \tag{2.11}$$

of the revolutionary motion around the fixed point. Thus, it is our aim to derive a Fokker-Planck equation describing the W-evolution. As a first step, we perform a further change of coordinates from (q, w) to (q, W). The transformed equations are

$$\dot{q}=\sqrt{2(W-V)}-\frac{\mu\tilde{\gamma}D}{(1+\tilde{\gamma})^2}[W+q-2(e^q-1)]$$
$$+\frac{\mu Q}{2}e^q+\sqrt{\mu Q}\,e^{-q/2}\Gamma(t)$$

$$\dot{W}=2\mu(W-V)\left(1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}e^q\right)+\frac{\mu Q}{2}(1-e^{-q})$$
$$-\frac{\mu\tilde{\gamma}D}{(1+\tilde{\gamma})^2}(e^q-1)[W+q-2(e^q-1)]$$
$$+(e^q-1)\sqrt{\mu Q}\,e^{-q/2}\Gamma(t) \tag{2.12}$$

where we note that, at variance with standard models, both variables are affected by noise. Equivalently, we can write the 2-d Fokker-Planck Eq. for the probability density $\tilde{P}(q, W, t)$

$$\frac{\partial\tilde{P}}{\partial t}=-\frac{\partial}{\partial q}\{\sqrt{2(W-V)}\tilde{P}\}$$
$$+\frac{\mu\tilde{\gamma}D}{(1+\tilde{\gamma})^2}\frac{\partial}{\partial q}\{[W+q-2(e^q-1)]\tilde{P}\}$$
$$+\mu\frac{\partial}{\partial W}\left\{2(W-V)\left[1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}e^q\right]\tilde{P}\right\}$$
$$+\mu\frac{\partial}{\partial W}\left\{\frac{\tilde{\gamma}D}{(1+\tilde{\gamma})^2}(e^q-1)[W+q-2(e^q-1)]\tilde{P}\right\}$$
$$-\mu Q\frac{\partial\tilde{P}}{\partial W}+\mu Q\frac{\partial^2\tilde{P}}{\partial q^2}$$
$$+2\mu Q\frac{\partial^2}{\partial q\,\partial W}[(1-e^{-q})\tilde{P}]$$
$$+\mu Q\frac{\partial^2}{\partial W^2}[e^{-q}(e^q-1)^2\tilde{P}] \tag{2.13}$$

We now assume $\tilde{P}(q, W, t)$ to factor-out as [6], [10]

$$\tilde{P}(q, W, t)=P_0(q, W)P(W, t) \tag{2.14}$$

where $P_0(q, W)$ describes the q-distribution in an approximate way as the contribution deriving from the deterministic oscillating motion, i.e., it is assumed to be proportional to $1/\dot{q}$

$$P_0(q, W)=\frac{N(W)}{\sqrt{W-V}}, \tag{2.15}$$

where $N(W)$ is the normalization factor

$$\frac{1}{N(W)}\equiv\oint\frac{dq}{\sqrt{W-V(q)}}. \tag{2.16}$$

By substituting (2.15) in (2.14) and then in (2.13), we see that the first term in the r.h.s. of (2.13) vanishes since the q-dependence disappears. As a consequence, only terms proportional to μ remain, indicating that the fast motion has been eliminated. By further conjecturing an approximate functional dependence of q on time t, with $\dot{q}\sim O(1)$, we note that the other derivatives with respect to q are equivalent to time derivatives. Therefore, being proportional to μ, they are negligible compared to the l.h.s. By finally integrating in q the remaining terms we obtain a closed equation for $P(W,t)$

$$\frac{\partial P}{\partial t}=-\mu\frac{\partial}{\partial W}[(Q-H(W)-F(W))P] +\mu Q\frac{\partial^2}{\partial W^2}[G(W)P] \tag{2.17}$$

where

$$\begin{aligned} H(W)&=N(W)\oint 2\sqrt{W-V}\left(1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}\mathrm{e}^q\right)\mathrm{d}q\\ F(W)&=N(W)\oint\frac{\tilde{\gamma}D}{(1+\tilde{\gamma})^2}\frac{\mathrm{e}^q-1}{\sqrt{W-V}}[W+q-2\mathrm{e}^q+2]\,\mathrm{d}q\\ G(W)&=N(W)\oint\frac{\mathrm{e}^{-q}(\mathrm{e}^q-1)^2}{\sqrt{W-V}}\mathrm{d}q. \end{aligned} \tag{2.18}$$

The relevance of (2.17) is twofold. On one side, it allows to describe in an accurate way the dynamical behavior of the probability distribution (including the long-time relaxation processes, here not considered); on the other hand, it leads to a simple expression for the stationary solution

$$P_{\mathrm{st}}(W)=\frac{N_0}{QG(W)}\exp\left[\int\frac{Q-H(W')-F(W')}{QG(W')}\mathrm{d}W'\right]. \tag{2.19}$$

The accuracy of expression (2.19) depends on the size of the two terms which have been neglected in the Fokker-Planck Eq. (2.13). As we have already mentioned, they depend on the ratio between the period of the oscillations and the time scale of the dissipative processes. Therefore, the accuracy of the adiabatic elimination not only controls the dynamics, but also the asymptotic probability distribution. In this respect the solution of (2.13) differs from the standard weakly damped oscillators with the noise source on the momentum, where the extra terms giving rise to the above mentioned difficulty do not appear. Moreover, integrals (2.16) and (2.18) are not easy to evaluate, thus we look for a perturbative expansion for small W-values. We will see that this provides a sufficiently good approximation for a large range of parameter values. Here, we limit to outline the appropriate procedure, directly giving the final results. As a first step a new integration variable $y\equiv q/\sqrt{2W}$ is introduced, together with the smallness parameter $\alpha\equiv\sqrt{2W}/3$. Next, we estimate the extrema y_1, y_2, by imposing that the kinetic energy $W-V(y)$ be 0. Such estimates and the integrands are all evaluated up to the fourth order in α, which corresponds to a second order in W. Finally, another change of variable is performed, namely, $x=2\frac{y-y_1}{y_2-y_1}-1$, which leads to the symmetric integration interval $[-1, 1]$. With these assumptions, the above integrals are easily computable and they are

$$\begin{aligned} \frac{1}{N(W)}&=\sqrt{2}\pi\left(1+\frac{W}{12}+\frac{W^2}{576}+\dots\right)\\ H(W)&=\left(1+\frac{\tilde{\gamma}D}{1+\tilde{\gamma}}\right)\left(W-\frac{W^2}{24}+\dots\right)\\ F(W)&=-\frac{\tilde{\gamma}D}{(1+\tilde{\gamma})^2}\left(W-\frac{W^2}{24}+\dots\right)\\ G(W)&=W+\tfrac{11}{24}W^2+\dots. \end{aligned} \tag{2.20}$$

Therefore, the stationary distribution is, up to second order terms,

$$P_{\mathrm{st}}(W)=N(1+\tfrac{11}{24}W)^{-(2+\frac{288}{121}K)}\mathrm{e}^{\frac{K}{11}W} \tag{2.21}$$

where

$$K\equiv\frac{1}{Q}\left[1+D\left(\frac{\tilde{\gamma}}{1+\tilde{\gamma}}\right)^2\right]. \tag{2.22}$$

Unfortunately, the sign of second order corrections is such that it yields a non-normalizable probability distribution. However, in the limit of small W- and large K-values, we can expand (2.21), proving that the distribution is well approximated by the simple exponential

$$P_{\mathrm{st}}=\beta\mathrm{e}^{-\beta W} \tag{2.23}$$

where $\beta=K+11/12$. In the next section we will see the role played by the corrections, comparing the theoretical expressions with the results of numerical simulations. Here, instead, we estimate the distribution of laser intensity, by referring to the simplified (2.23). By substituting (2.23) and (2.15) into (2.14) we find the joint distribution

$$\tilde{P}_{\mathrm{st}}(q,W)=N\frac{\mathrm{e}^{-\beta W}}{\sqrt{W-\mathrm{e}^q+q+1}}. \tag{2.24}$$

By means of (2.9) and (2.11), the probability density (2.24) can be expressed in terms of the population inversion w and of the intensity I,

$$I\equiv\frac{|E|^2}{\langle|E|^2\rangle} \tag{2.25}$$

rescaled to its mean value D. By further integrating in w, we obtain the distribution of the sole intensity

$$\bar{P}(I)=\frac{(\beta)^\beta}{\Gamma(\beta)}I^{\beta-1}\mathrm{e}^{-\beta I} \tag{2.26}$$

where Γ is the Euler gamma function. It is interesting to note that, in the limit D tending to 0, the average intensity vanishes as it should, while the pseudo energy $W(\sim\beta)$ diverges as $1/D$. The general expression for moments, M_n, and cumulants, K_n, of distribution (2.26) is

$$M_n = \beta^{-n} \frac{\Gamma(\beta+n)}{\Gamma(\beta)}$$
$$K_n = (n-1)!\, \beta^{-n+1}. \tag{2.27}$$

A convolution with a Poissonian distribution allows to find the final expression for the photon-counting distribution. However, still referring to (2.26) we see that the maximum of the probability density is attained at $I = 1 - 1/\beta$, very close to the mean value, 1, since $\beta \gg 1$. Moreover, we observe that the relative variance (second cumulant) K_2 is very small, and higher-order cumulants are increasingly smaller, so that the distribution (2.26) is well approximated by a Gaussian with variance $1/\beta$. It is also instructive to make a comparison with the analogous results obtained for class-A lasers, where, we recall, the relative variance scales as $\langle I \rangle^{-2}$. In the present case, from (2.22) and (2.6), we find

$$K_2 \propto \left[\langle I \rangle - \langle I \rangle^2 \left(\frac{\tilde{\gamma}}{1+\tilde{\gamma}}\right)^2\right]^{-1} \tag{2.28}$$

which exhibits a weaker dependence on the intensity for D-values of order $O(1)$.

3. Numerical simulations

Here, we compare the analytical results derived in the previous section with numerical simulations of the nonlinear Langevin Eq. (2.10). The comparison is done for different values of the pump parameter D, which is the quantity most controllable in experiments. Beside its explicit presence in the starting model (2.10), it also enters the definitions of μ and Q, (2.4) and (2.6) respectively, and the time rescaling. The other parameters are chosen with reference to the real case of CO_2 lasers where, since $\mu(D=1), Q(D=1) \sim 10^{-2}$, our analysis fully applies for pump values of order $O(1)$. Very close to threshold ($D \sim 0$), instead, μ diverges, indicating that the motion in the Toda potential becomes more and more overdamped. Simultaneously, the amplitude of the stochastic force diverges as $D^{-3/2}$. The approach devised by Risken in [2] applies to such a limit region. Therefore, we expect a crossover behavior to separate the regions of applicability of the two dynamics, and it is interesting to estimate the order of magnitude of D where it occurs. The lower boundary of the D region where our method is expected to be in a qualitative agreement with experiments is roughly estimated as the value where either the losses or the stochastic forces become non-negligible. In a mathematical language it is the maximum between D_1 and D_2, where D_1, D_2 are defined as $\mu(D_1)=1, Q(D_2)=1$. Still referring to a CO_2 laser, the validity region extends very close to threshold: indeed the limit value is $D_2 \sim 10^{-2}$. In the opposite limit, $D \to \infty$, the losses again diverge (as $D^{1/2}$) leading to an overdamped motion, whereas the noise amplitude decreases as it should, since the quantum effects vanish. Therefore, we are not interested in such a region where a statistical approach is no more required. The nonlinear Langevin Eq. (2.10) for the logarithm q of the intensity and the population inversion w have been numerically inte-

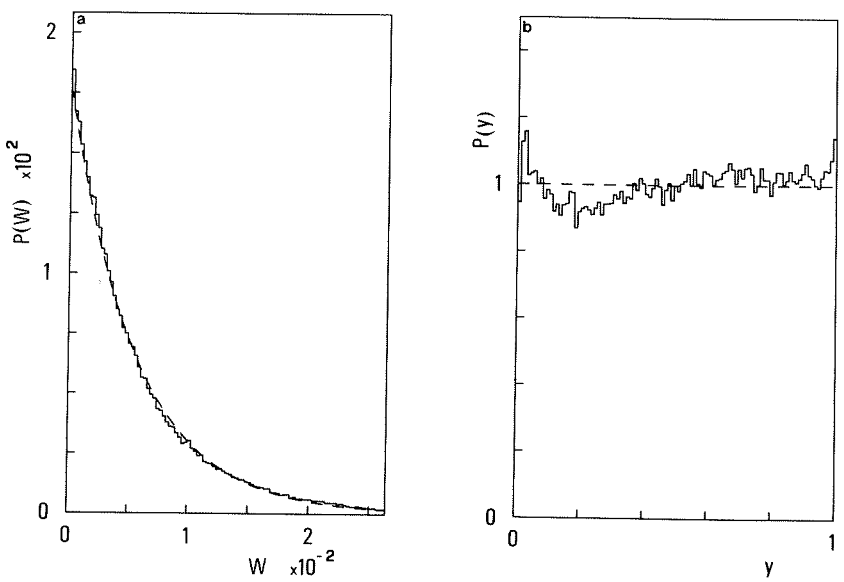

Fig. 1. a Theoretical probability distribution of the pseudo-energy W (dashed line) compared with numerical results (hystogram); **b** the same comparison by using $y = \exp(-\beta W)$. The pump parameter D is equal to 1, and the noise strength $Q = 10^{-2}$

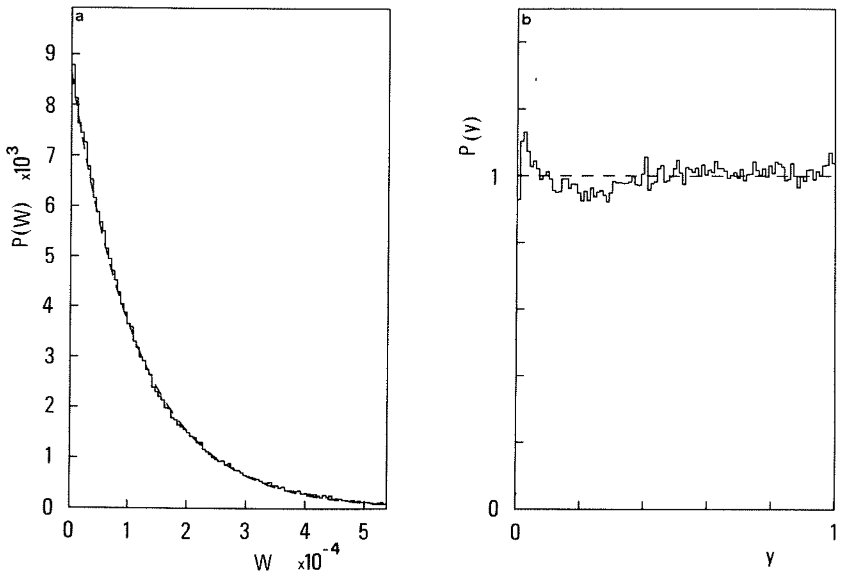

Fig. 2a and b. Same as in Fig. 1, with $D=10$

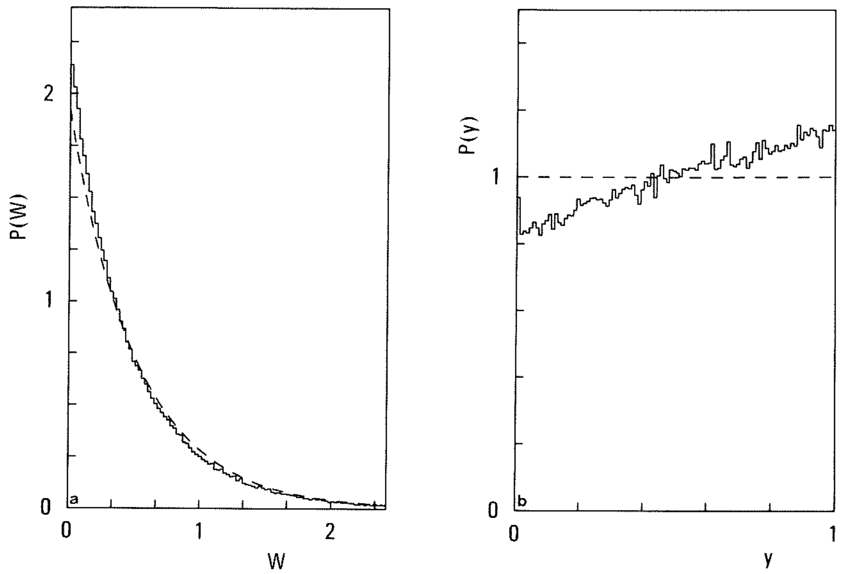

Fig. 3a and b. Same as in Fig. 1, with $D=10^{-2}$

grated by means of a fourth order Runge-Kutta method (integration step equal to 10^{-2}) and adding, at each step, a random kick with Gaussian distribution and a suitable variance which yields the expected diffusion coefficient in the associated Fokker-Planck equation In order to emphasize the difference between the expected distribution and the numerical results we have sometimes used the auxiliary variable $y=\exp(-\beta W)$. In fact, the energy distribution should become, if (2.23) were correct, a flat line between 0 (infinite energy) and 1 (zero energy). Therefore, the deviations from an exponential behavior can be immediately detected. All the simulations have been performed integrating for a time $T=2\cdot 10^5$, with a sampling time equal to 1. In Fig. 1a the probability distribution $P(W)$ is plotted for $D=1$, showing a very good agreement with the theoretical expression (2.23). The nice accuracy is also confirmed in Fig. 1b, where $P(y)$ is drawn. The next simulation (Fig. 2a, b) indicates that still for $D=10$ the theoretical prediction is close to the numerical results. On the other side, an integration for $D=10^{-2}$ suggests a good agreement (see Fig. 3a, b). It is worth to note that such a reasonable agreement occurs despite W-values are no longer negligible with respect to 1 and an expansion in a W-power series is not justified. Furthermore, we have performed one more simulation with $D=1$, but a larger value of $Q(=0.1)$ to control the sensitivity of

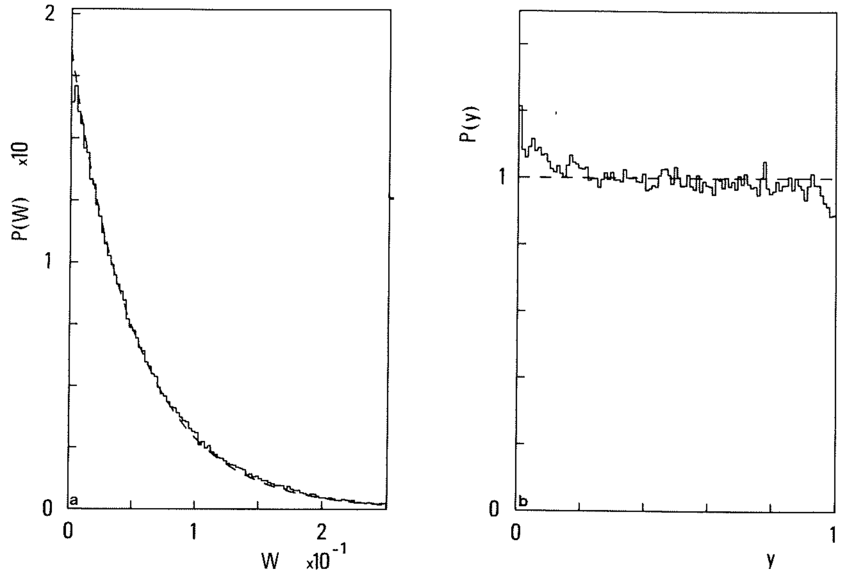

Fig. 4a and b. Same as in Fig. 1, with $Q=0.1$

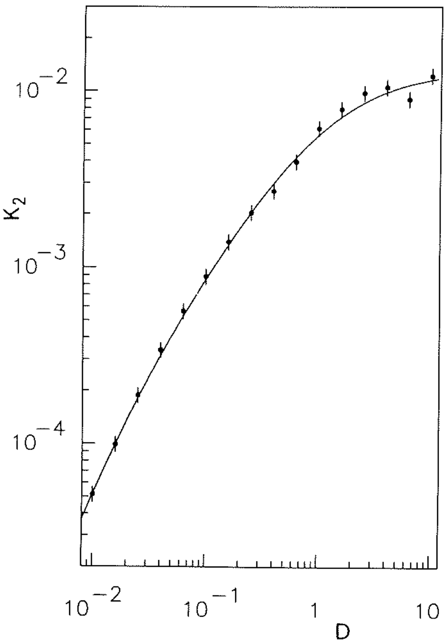

Fig. 5. Variance of field intensity versus *D*. The continuous curve represents the theoretical expression, while the dots refer to numerical simulations (with a statistics of 5000 independent data). Three orders of magnitude in *D* are investigated

the method to the strength of the noise. The results in Fig. 4a, b again confirm the accuracy of the theoretical approximations.

Finally, to make a quantitative comparison between theory and numerical experiments, we computed the variance of the field intensity, which turns out to be the most relevant physical variable. This has been done, by exploring a range of three orders of magnitude in *D*. The results, obtained with a statistics of $5 \cdot 10^3$ independent data, are plotted in Fig. 5. They are in agreement, within the statistical uncertainty (denoted by the vertical bars), with the theoretical results. It is useful to compare such a dependence of the variance on *D*, with the corresponding behavior for class-A lasers. To do that, recall the definition of the pump parameter *a* in terms of the actual parameters

$$a=\frac{2}{\mu\sqrt{Q(1)}}\sqrt{\frac{\tilde{\gamma}}{1+\tilde{\gamma}}D} \tag{3.1}$$

For a CO_2 laser $a \sim 2 \cdot 10^3 D$. At variance with class-A lasers, where the width of the distribution already saturates at *a*-values around 10 ($D \sim 10^{-2}$), here we observe a saturation only for $D > 10$. Finally, the proportionality constant between *a*, and *D* allows to underline the separation between the range of applicability of our method ($D \sim 1$, roughly speaking) and that of [3] ($a \sim 1$). As is known [2], experiments span over the wide range $0 < D < 10$, accounted for by our results.

Therefore, the present approach, based on a further reduction of the number of variables (beyond the elimination of atomic polarization) from 2 to 1 for the pseudo-energy proves to be very powerful in the statistical description of class-B lasers. An approximate analytic expression for the stationary probability distribution has been derived yielding very accurate results. As a result, a simple analysis of the long-time relaxation properties of the probability distribution is made possible, as well.

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9. To our knowledge, the only exception is represented by a paper of A. Fernández: Phys. Rev. A **32**, 3070 (1985), where a chemical instability is considered
10. The meaning of this factorization consists in attributing the time dependence exclusively to the slow variable W, whereas the fast one, q, thermalizes to its equilibrium set controlled by W. Also the reduction from the whole set of Maxwell-Bloch Eq. (2.1) to the 2-d system (2.8) could be formulated in terms of a factorization of probabilities

P. Paoli, A. Politi, F.T. Arecchi
Istituto Nazionale di Ottica
Largo Enrico Fermi, 6
I-50125 Arcetri-Firenze
Italy