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Statistical Properties of a Class of Deterministic Cellular Automata.

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Abstract. – We have studied a class of three-state cellular automata (CA) which can develop complex patterns. From the probability distribution of *substrings* we are able to determine the *multifractal* nature of this CA and to relate it with some of its *grammatical* properties.

Understanding spatio-temporal complexity in dynamical systems is one of the most important challenges in physics and mathematics [1]. Turbulence is still a paradigm of this behaviour. In recent years some routes to temporal chaos in strongly confined systems have been understood. The main effort now is to describe the behaviour of weakly confined, or spatially extended, systems. Among them one can cite the recent evidence of strong disorder in one-dimensional partial differential equations (PDE), in particular the Kuramoto-Shivashinsky equation [2]. A model study of PDEs can be done on spatially coupled nonlinear maps [3], such as the logistic map or the circle map, to see how the information loss in an oscillator propagates on a spatial lattice in course of time [4].

By far the simplest dynamical system is a cellular automaton (CA) [5]. It consists in a lattice of cells each having a discrete number of states and evolving through them via discrete time steps. The rules of evolution can be deterministic or probabilistic. Here we will concern only with deterministic CA. These are classified according to the number of states each site can take, the number of neighbours that determine the evolution of a site and the rule itself. For example, one can study a 3-state CA, with a 2-neighbour interaction. From the values of these three sites one determines the evolution of the central site. (Of course, the rule must be translationally invariant.) The possible number of rules in this case is $3^{(3^3)}$.

Among these rules we chose some with a physical motivation. The selected CA were proposed by Oono and Kohmoto [6] in order to mimic the reaction-diffusion equation on a

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discrete space and time. They consist of 3-state CA ruled by an equation as

$$x_{t+1}^{n} = F\left[x_t^n + \frac{\alpha}{2}(x_t^{n-1} - 2x_t^n + x_t^{n+1})\right], \qquad (1)$$

where x can assume the values 0, 1 and M, the superscripts denote the space site, the subscripts the time position and $F(\xi) = 1, 0$ or M depending on whether $\xi \geq 1.5$, or $0.5 \leq \xi < 1.5$, or $\xi < 0.5$, respectively.

Different configurations can be observed for different values of the parameters, $\alpha \in [0, 1)$ and $M \in \boldsymbol{N}$, namely, «periodic», «solitonic» and «turbulent» patterns [6].

In general, even in the case of a deterministic CA it is very difficult to predict the evolution simply from its rule [5], and for this purpose one must recur to simulation.

Trying to establish the statistical properties of CA (1), one has to account that they are «irreversible», that is a significant class of initial conditions cannot be recovered in the subsequent evolution [7], and from some time on, the motion remains confined on a subset of the initial conditions. To account for this, we eliminate the transient part of the evolution. The infinite spatial extension of the CA is simulated by keeping the number of iterations less than the number of sites multiplied by the estimated speed of propagation of perturbations.

The aim of the present work is to study the statistical properties of the «turbulent» phase for $\alpha = 0.3$, $M = 4$ (we call $A3M4$ such CA), and to evaluate the complexity of the associated invariant measure. In so doing, we have generated the $A3M4$ CA taking a one-dimensional system of $2.5 \cdot 10^5$ sites and following its evolution for more than 10^5 iterations.

Making use of a method introduced by Grassberger [8], we have represented in a plane the values (base 3) of two strings of 10 pixels separated by a pixel and codes as decimal numbers in the interval [0, 1]. The first 100 iterations have been excluded to avoid transient effects. These neighbour strings must show a maximum correlation (if any).

In fig. 1a) we have collected the values of 10^5 iterations for the $A3M4$ CA. This «attractor» displays a clear fractal structure, which can be thought of as a necessary consequence of the «grammatical» rules. As shown by Oono and Yeung [6b], the $A3M4$ rule eliminates the sequences $1M$ and $M1$ during its evolution. (Note that the particular value

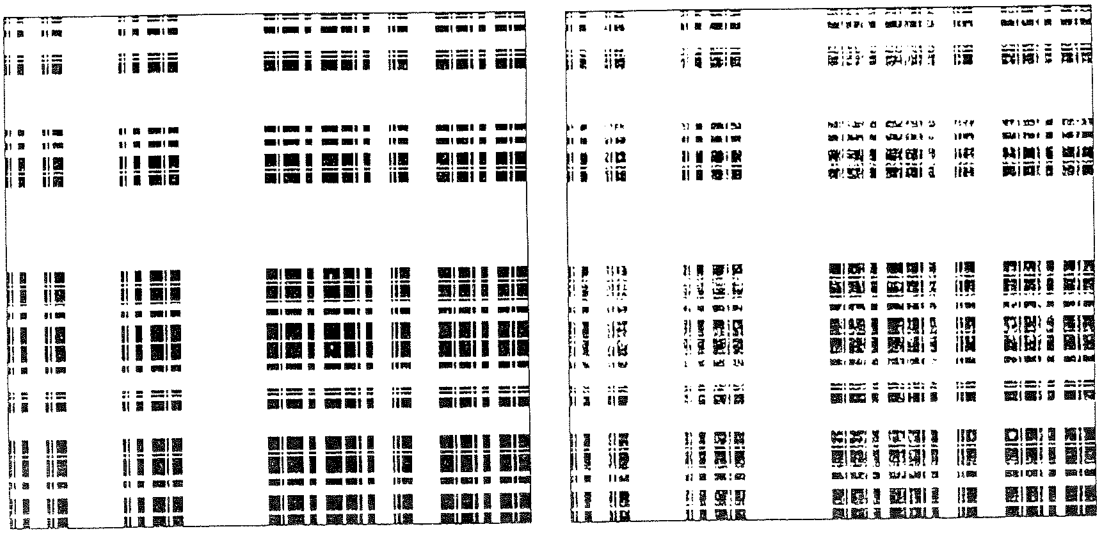

Fig. 1. – a) Attractor of the $A3M4$ CA. b) Attractor generated by all the number in the square [0, 1][0, 1] encoded in base three which do not contain the sequences 12 and 21.

$M = 4$ is used only to define the CA rule, but for the purpose of coding the string it is better to identify M with 2.) This suffices to create a «generalized» Cantor set. For the sake of comparison we show in fig. 1*b*) all the 2D vectors whose components base 3 are deprived of the sequences 12 and 21. Applying formal language methods [5, 9] it is easily proved that only these sequences are eliminated in the evolution of the *A3M4* CA.

From this observation we can evaluate analytically the fractal dimension $\mathcal{D}_0$ of such a generalized Cantor set. Its evolution can be represented by a tree as shown in fig. 2. This

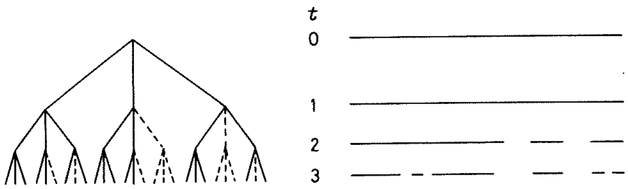

Fig. 2. – *a*) Tree structure of the sequences in the *A3M4* CA. *b*) Cantor-like partition of the unit interval in the first three time steps.

shows a structure slightly different from the classical Cantor set. The number of branches of a subtree at each step can be obtained by the following relation:

$$\begin{pmatrix} T \\ D \end{pmatrix}_{t+1} = \begin{bmatrix} 1 & 1 \\ 2 & 1 \end{bmatrix} \begin{pmatrix} T \\ D \end{pmatrix}_t , \tag{2}$$

where T_t and D_t stand for the number of subtrees with three and two branches at the t-th time step, respectively. The solution of this equation with the initial conditions $T_0 = 1$ and $D_0 = 0$ is

$$\begin{pmatrix} T \\ D \end{pmatrix}_t = \begin{pmatrix} 1/2 \\ 1/\sqrt{2} \end{pmatrix} (1+\sqrt{2})^t + \begin{pmatrix} 1/2 \\ -1\sqrt{2} \end{pmatrix} (1-\sqrt{2})^t . \tag{3}$$

The number of branches $\mathcal{N}_t$ at each time step must be $\mathcal{N}_t = (3T_t + 2D_t)$ and, therefore, the fractal dimension of this set is

$$\mathcal{D}_0 = \lim_{t \to \infty} \frac{\ln \mathcal{N}_t}{\ln 3^{t+1}} = \frac{\ln(1+\sqrt{2})}{\ln 3} = 0.8022608\ldots \tag{4}$$

Considering 10^8 strings of 10 pixels on the pattern obtained by the evolution of the *A3M4* CA we measure $\mathcal{D}_0 = 0.80226\ldots$. For the sake of precision we used an optimized method [10], instead of a simple box counting. The agreement between calculated and measured $\mathcal{D}_0$ supports the conjecture that the fractal character of this attractor comes only from the elimination of the sequences 12 and 21. However, in order to fully characterize the attractor, we must study the probability distribution of the allowed strings. From this distribution it is easy to calculate the Shannon entropy of strings of N pixels $\mathcal{H} = -\sum_{\langle s \rangle} p_N \ln p_N$, where $\langle s \rangle$ is the complete set of these strings. Therefore, the quantity $h_N = \mathcal{H}_{N+1} - \mathcal{H}_N$ represents the additional information needed to predict s_{N+1} when the string $s_1, \ldots, s_N$ is known. This quantity is called the *N-th–order block entropy* [8] and it is a monotonically decreasing function of N. Its value saturates at $h = h_n$ for all $n \geqslant N$ when the sequence is a n-th-order Markov chain [11]. In fig. 3 the variation of the block entropy h_N with N is shown. One sees that a saturation value is attained for $N \geqslant 2$.

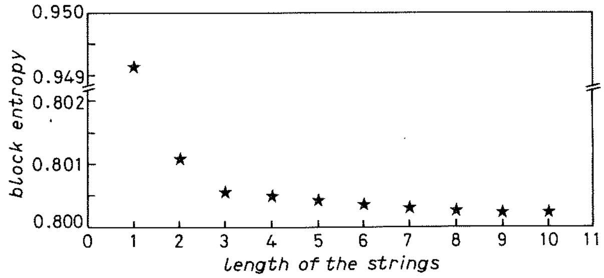

Fig. 3. – Block entropies from the *A3M4* CA.

As stressed by Grassberger [11] a slow decay of h_N for $N \to \infty$ is linked to a high *complexity* of the pattern. A quantitative measure of this complexity is the so-called *effective measure complexity* (EMC), defines as

$$\mathrm{EMC} = \sum_{N=1}^{\infty} N(h_{N-1} - h_N) = \sum_{N=0}^{\infty} (h_N - h), \tag{5}$$

where $h = \lim_{N \to \infty} h_N$ is the block entropy of an infinite string. The result in fig. 3 allows one to conclude that although the *A3M4* CA develops nontrivial patterns, its statistical complexity is relatively small, because $\mathrm{EMC} \simeq 0.15$. This is significantly smaller than the value $\mathrm{EMC} \to \infty$ estimated by Grassberger [11] for the CA rule 22 (in the notation of Wolfram [5]) of for the binary sequence generated at the accumulation point of subharmonic bifurcations in the logistic map.

The conjecture that the longest Markov chain generated by the statistics is of order two can be checked assuming the following *ansatz*:

$$p(s_1, s_2, s_3, \ldots, s_n) = p(s_n | s_{n-1}) \ldots p(s_2 | s_1)\, p(s_1), \tag{6}$$

where $p(\,|\,)$ denotes the conditional probability, defined as $p(s_i|s_j) = p(s_i, s_j)/p(s_j)$, in terms of the joint probability $p(s_i, s_j)$. After introducing the values of conditional and single site probability obtained from the statistical distribution one can compare them with those obtained directly by box counting of all the strings of a given length. The differences are very small, but in order to have a more quantitative estimate of this feature it is convenient to analyse the *multifractality* [12, 10] of the statistical distribution of that attractor. This is obtained by means of the Renyi dimensions $\mathscr{D}_q$ [10] with the partition size l scaled progressively as 3^{-N}.

In our case it is natural to use the box-counting method to determine these dimensions because the chosen length of strings gives a natural partition of the interval. It is useful to introduce the *spectrum of singularities* $f(\alpha)$ [10] in the probability distribution, $p_i^q \propto \varepsilon^{q\alpha_i}$. The function $f(\alpha)$ for the attractor of the *A3M4* CA is represented in fig. 4.

The full line corresponds to the multifractal function obtained simply from the statistics of the strings of 10 pixels, while the dashed one has been obtained by means of the *ansatz* (6), after introducing the fractal dimension of the generalized Cantor set and the probabilities $p(s_i)$ and $p(s_i, s_j)$ obtained for strings of one and two pixels, respectively. The best agreement is obtained for the positive dimensions (those physically more significant), which correspond to the left part of the $f(\alpha)$ function, while for the negatives ones the agreement is worse. This fact is not surprising because the main discrepancies must lie in the less dense regions of the probability distribution. Another important feature in fig. 4 is that the

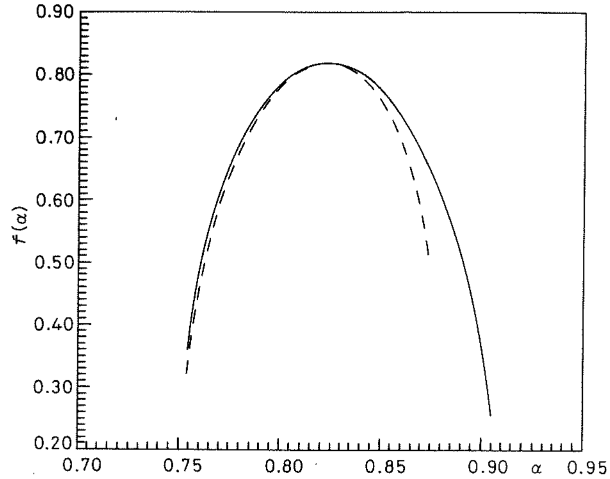

Fig. 4. – $f(\alpha)$ function. Probability distribution of strings of 10 pixels in the *A3M4* CA (full line) and from a factorized probability distribution (dashed line) (see the text).

spectrum of singularities α lies in a very small interval, which indicates that the probability distribution is quite uniform.

In summary we have demonstrated that the CA proposed by Oono and Kohmoto to mimic chemical turbulence does not have a high complexity from a physical point of view, that is in terms of an EMC [11], even though in the ordinary algorithmic sense [13] the spatial pattern generated is rather complex. The *multifractal* structure of the attractor generated by the *A3M4* CA can be roughly reproduced using the main *grammatical rules* generated by this CA to calculate the fractal dimension and a simple *ansatz* to generate the statistical distribution.

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