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Phase diffusion in chaotic laser dynamics

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The role of phase dynamics in laser systems is investigated. A detailed analysis of a bidirectional ring laser shows that, besides a trivial shift of the average frequency, a diffusion originated by the chaotic motion can be detected. In this case, even if the evolution of the average phase is removed by a suitable change of reference, the fractal dimension of the attractor remains unchanged, at variance with the ordered motion, where the same choice would reduce the dimension by one.

So far, chaotic laser dynamics has been investigated mainly in terms of the field intensity.[1] Only very recently, attention has been focused on the phase dynamics as well.[2] In principle, according to the embedding theorem,[3] the phase is as good a variable as the amplitude for reconstructing the underlying strange attractor and so it is not surprising that phase evolution contains relevant information. In general, the dynamics of laser systems can be factored out as the closed evolution in a lower-dimensional subset of the state space, plus the externally driven evolution of the remaining degrees of freedom (typically a phase). In the case of a single-mode laser with a homogeneous gain line, the dynamics are ruled by five equations coupling two complex variables (field amplitude and polarization) and a real one (inversion). However, this model can be reduced to a four-dimensional set of differential equations, linking population inversion, field and polarization moduli, and the phase difference between polarization and electric field. The evolution of the fifth variable (field or polarization phase) is determined by the other degrees of freedom without any feedback.[4] It should then appear correct to conjecture that the phase does not contribute to the "complexity" of the dynamical behavior of the entire system, that is, the dimension estimate obtained from the phase data is equal to that from any other variable. It is the aim of the present Rapid Communication to show that this assumption does not hold in the case of chaotic motion because of phase diffusion.

In this work we have analyzed the behavior of a bidirectional ring laser since its phase dynamics are richer than, though similar to, that of the single-mode case. We refer to class-B lasers characterized by a small-population decay rate, so that the medium polarization can be adiabatically eliminated. The model, written in polar coordinates in a suitable frame rotating at a frequency ω_R, is[5]

$$\begin{aligned}
\dot\rho_1 &= w\rho_1+\rho_2 d_1\cos(\psi)+\rho_2 d_1\delta\sin(\psi)\,,\\
\dot\rho_2 &= w\rho_2+\rho_1 d_1\cos(\psi)-\rho_1 d_1\delta\sin(\psi)\,,\\
\dot w &= A-(\rho_1^2+\rho_2^2)-\epsilon[w(1+\rho_1^2+\rho_2^2)+\rho_1\rho_2 d_1\cos(\psi)]\,,\\
\dot d_1 &= -\rho_1\rho_2\cos(\psi)-\epsilon[d_1(1+\rho_1^2+\rho_2^2)+w\rho_1\rho_2\cos(\psi)+d_2\rho_1\rho_2\cos(\Delta\phi-\Delta\eta_2)]\,,\\
\dot\phi_1 &= -\delta w+\frac{\rho_2 d_1}{\rho_1}[\sin(\psi)-\delta\cos(\psi)]\,,\\
\dot\phi_2 &= -\delta w-\frac{\rho_1 d_1}{\rho_2}[\sin(\psi)+\delta\cos(\psi)]\,,\\
\dot\eta_1 &= -\frac{\rho_1\rho_2}{d_1}\sin(\psi)-\epsilon\frac{\rho_1\rho_2}{d_1}[w\sin(\psi)+d_2\sin(\Delta\phi-\Delta\eta_2)]\,,\\
\dot d_n &= -\epsilon[d_n(1+\rho_1^2+\rho_2^2)+d_{n-1}\rho_1\rho_2\cos(\Delta\phi-\Delta\eta_n)+d_{n+1}\rho_1\rho_2\cos(\Delta\phi-\Delta\eta_{n+1})]\,,\\
\dot\eta_n &= -\epsilon\frac{\rho_1\rho_2}{d_n}[d_{n-1}\sin(\Delta\phi-\Delta\eta_n)+d_{n+1}\sin(\Delta\phi-\Delta\eta_{n+1})]\,.
\end{aligned}\tag{1}$$

Here, ρ_1,ρ_2 are the field amplitudes of the two counterpropagating waves and ϕ_1,ϕ_2 are the respective phases. The frequency ω_R is chosen such that $\theta=-\delta$ (pulling condition[1]) and the time is expressed by the adimensional variable $\tau=\sqrt{\gamma\kappa}t$, γ and κ being the decay rate of population and field amplitude, respectively. The population inversion has been expanded in Fourier modes, of amplitude D_n. Following Ref. 6 the zeroth mode real amplitude D_0 has been rescaled as $w=(D_0-1)/\epsilon$, where ϵ is the smallness parameter,

$$\epsilon\equiv\left(\frac{\gamma}{\kappa}\right)^{1/2}. \tag{2}$$

The nth Fourier complex amplitude D_n is described by an amplitude d_n, rescaled by a factor ϵ, and by a phase η_n. Finally the quantity ψ is defined as

$$\psi \equiv \phi_2 - \phi_1 - \eta_1 , \tag{3}$$

and $\Delta\eta_n = \eta_n - \eta_{n-1}$, $\Delta\phi = \phi_2 - \phi_1$.

One can prove[5] that the higher-population harmonics scale as $d_n \propto \epsilon d_{n-1}$. Thus, in first approximation, the contribution of the second harmonic can be neglected in the equations for d_1 and η_1. Thus the system reduces to seven equations, five of which (those for ρ_1, ρ_2, d_1, w, and ψ) are closed onto themselves,[7] while the remaining phase variables can be simply obtained by quadrature. This is clarified in the block diagram shown in Fig. 1. The dynamical system of Eq. (1) is seen as seven main variables loosely coupled to a bath of all d_n, η_n $(n>1)$. By splitting the phases into a main one ψ and two extra ones (e.g., ϕ_1 and ϕ_2), the seven equations can be represented as a block of five variables with a one-sided coupling with the two extra phases. However, these latter ones are recoupled to the five main variables via the bath. As a matter of fact this recoupling is very weak due to the scaling law of the harmonics so that it can be disregarded in the present analysis (see later, however, for an analysis of the effect of the back coupling on the estimate of the dimension). Therefore, we are in the presence of a situation similar to the single-mode case.

When the laser exhibits a steady-state intensity, model (1) can, nevertheless, display a rotation due to a constant drift of the phases. As pointed out in Ref. 2(b) in the single-mode case, such a trivial contribution can be removed by choosing the frequency ω_R of the reference frame equal to that of the lasing state. Accordingly, the phase dynamics confirms the steady-state nature of the asymptotic solution.

An analogous conclusion can be drawn in the presence of a periodic self-pulsing regime. If the original frequency ω_R is chosen equal to the time average of the instantaneous laser frequency, no secular contribution is added by integrating one of the phase equations. Thus, besides a trivial contribution which can be canceled out by an appropriate choice of the reference frame, the phase conveys full information on the nonlinear dynamics as any other variable, and hence a string of phase data can be chosen as a relevant set to reconstruct the attractor by an embedding technique.

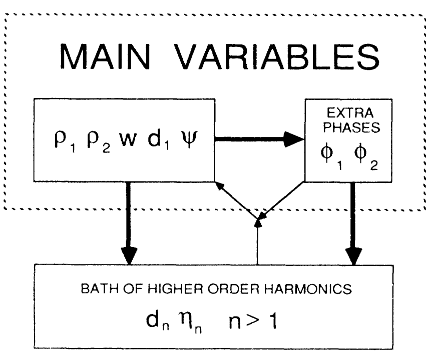

FIG. 1. Block diagram of Eq. (3). The thickness of the arrows is proportional to the coupling strength. The five main variables, while strongly driving the other two blocks, are only weakly recoupled via the bath. The arrangement of the thin arrows is intended to emphasize the structure of the recoupling onto the d_1 equation.

The above picture drastically changes as soon as we pass to analyze a chaotic regime. A simple inspection of the phase equations shows that the right-hand side is independent of the phase itself, so that the associated Lyapunov exponent is equal to zero. Therefore, according to the Kaplan-Yorke conjecture,[8] the effect of the phase is to increase the dimension of the whole attractor by one. The problem is formally equivalent to the limit case of a filtered chaotic signal with a low-frequency cutoff set equal to zero, a situation where the dimension increase has indeed been observed.[9]

At variance with the ordered regime, where a suitable change of coordinates permits the elimination of the phase drift, thus lowering the dimension by one, in the chaotic case this is no longer possible. In fact, since no dissipation is associated with the phase motion and the source term is essentially stochastic, we expect that phase diffusion would occur and that it would fill, in a nontrivial way, the state space along the new direction. The conclusion is that the phase not only cannot be eliminated, but it carries new relevant information on the evolution of the laser system.

In the absence of chaos, reconstruction of the attractor through a temporal series of either one of the five variables, or one of the two extra phases, yields the same dimension. However, when chaos is responsible for diffusion of the extra phases, the two reconstructions are no longer equivalent, indeed, an embedding of a temporal series of one of the phases yields an estimate of the attractor dimension higher by one than that obtained by analyzing one of the five main variables.

We have numerically integrated the model for different detuning values with a fixed scaling parameter $\epsilon = 0.01$, and pump strength $A = 4$ (reasonable numbers for a CO_2 laser). A typical behavior of the field phase ϕ_1 is shown in Fig. 2, for $\delta = 0.6$. Besides an average phase drift ($\langle\phi_1\rangle = 0.02t$, in this case), a linear diffusion of the variance is observed as reported in Fig. 2. Such behavior is

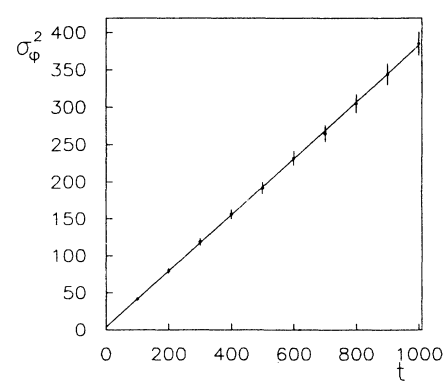

FIG. 2. Variance of the field phase ϕ_1 as a function of time. Parameter values $A = 4$, $\delta = 0.6$, and $\epsilon = 0.01$.

analogous to the diffusion in the presence of a stochastic force. In our case, the chaotic motion in the five-dimensional subspace provides the "random" source for the phase evolution. It is precisely such diffusion that prevents the phase from reproducing the dynamical behavior of the reduced subspace. This means that a reconstruction of the attractor from the phase data yields an increase of dimension by one.

By symmetry reasons, the phase ϕ_2 behaves as ϕ_1. The remaining two phases, η_1 and ψ, show a similar diffusive phenomenon with zero mean velocity. Indeed, the grid of population inversion can fluctuate back and forth, but there is no reason for a preferential drift in one direction.

Finally, we should consider more accurately the effect of the back coupling of the extra phases onto the five main variables. These are an independent block only if we truncate Eqs. (1) to the first harmonic. If, instead, we choose to consider some of the higher-order harmonics then only one phase continues to remain uncoupled to all the other variables, just as in the single-mode case. By the embedding theorem this means that we should obtain exactly the same dimension by considering whichever variable we may like, apart from the uncoupled phase; however, while this is correct in principle it can be very difficult to prove in practice. Indeed, the effect of the extra phases on the other "important" variables can be represented as a small noise which is added to the uncoupled equations. This noise tends to spread the orbit, thus giving rise to the increase of the attractor dimension, but it is opposed in this spreading action by the contraction rate towards the attractor's stable manifolds. Whence the added width results from the balance between these two counteracting effects, and if the stable manifolds are strongly contracting this added width can be so small that one needs to go to very small distances for its detection.

This can be easily seen in the simple model

$$
\begin{aligned}
x_{n+1} &= \tfrac{1}{3} x_n + \tfrac{2}{3} [2y_n] + \epsilon z_n \,, \\
y_{n+1} &= \mathrm{Mod}_1(2y_n) \,, \\
z_{n+1} &= z_n + (-1)^n (x_n + y_n) \,,
\end{aligned}
\tag{4}
$$

where the brackets indicate the integer part. The x,y equations reduce to the baker map in the limit $\epsilon = 0$, while z_n plays the role of the extra phases. To determine the width of the spreading due to the presence of z_n, one can approximately consider the first of Eqs. (4) as a Langevin equation for x_n with a damping coefficient λ_- (the negative Lyapunov exponent of the baker map) and a noise with diffusion constant $Q \equiv \langle \Delta z^2 \rangle / n$. The width of the resulting distribution of x_n,

$$
\bar{\delta} = \epsilon \left(\frac{Q}{2|\lambda_-|} \right)^{1/2} , \tag{5}
$$

is the length scale that one has to reach in order to determine the correct attractor dimension. We have numerically tested this hypothesis for $\epsilon = 0.05$. The Lyapunov exponents can be computed analytically while the diffusion constant has been determined numerically. The resulting length scale is $\bar{\delta} \simeq 0.03$ and the Lyapunov dimension is 2.62. Finally, we have computed the attractor information dimension D_1 via embedding technique on 250000 values of $u_n = x_n + y_n$ with the nearest-neighbor method.[10] The results are shown in Fig. 3. This is a plot of the dimension versus the decimal logarithm of the length scale examined. One can see that the correct attractor dimension is estimated only at distances of order $\bar{\delta}$, in agreement with our hypothesis.

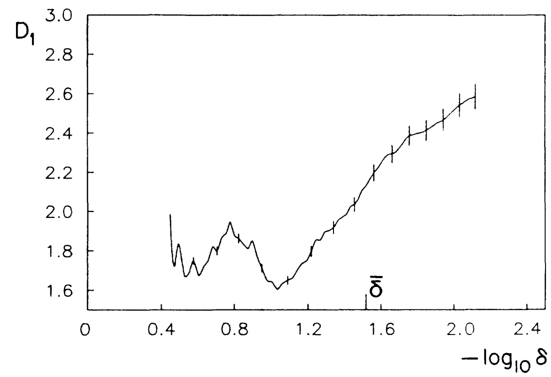

FIG. 3. Coarse-grained dimension of map (4) vs the decimal logarithm of the length scale examined, δ. $\bar{\delta}$ is the theoretical estimate of the length scale needed to observe the correct asymptotic dimension (in this case equal to 2.62).

In the case of the bidirectional ring laser the diffusive term is the last addendum in the equation for d_1,

$$
-\epsilon d_2 \rho_1 \rho_2 \cos(\Delta\phi - \Delta\eta_2) \,.
$$

From the numerical estimate of the diffusion velocity of this term (3.9×10^{-6} in the most favorable case found) and the order of the negative Lyapunov exponents of the model, one finds that the length scale associated with the increase of dimension is so small as to be practically unobservable.

We conclude by remarking that the phase dynamics convey enough information to lead to an unavoidable increase of the fractal dimension in the presence of chaotic motion. In the particular case of bidirectional ring lasers only one of the extra phases, ϕ_1 and ϕ_2, is coupled back with the intensity [as shown in the fourth expression of Eqs. (1)]. Thus the dimensional increment is one, as in the example of map (4), while the second phase has no effect whatsoever.

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