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Reduction of Variables in a Bidirectional Ring Laser.

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Abstract. – The reduction to a finite-dimensional model is discussed in the case of a class-*B* bidirectional ring laser. A semi-analytical estimate of the active degrees of freedom is proposed. As a result, the usual truncation to 7 equations is accurately checked.

The discovery of deterministic chaotic motion has opened the problem of deriving the simplest (least dimensional) model capturing the relevant features of a generic dynamical behaviour.

Such a problem is intimately related to the adiabatic elimination of fast variables proposed by Haken [1]. In fact, fast-decaying modes which relax toward a «local» equilibrium position are eliminated, being slaved by the slower modes. More recently, centre manifold theory [2] gave a rigorous mathematical base to this method, providing a powerful technique to determine a series expansion of the asymptotic manifold where the motion takes place. As a result, whenever two well-separated time scales can be detected in a model, it is possible to introduce a smallness parameter (namely, the ratio of such scales) controlling the convergence of the series expansion.

However, the final problem of extracting the simplest model which reproduces the chaotic features of the starting system is not exhausted by a centre manifold theory. In fact, we may expect to find general cases where the relaxation towards the «weakly» stable manifold is not sufficiently fast and the series expansion does not converge.

This more general situation is covered in principle by the inertial manifold theory [3], where the problem is approached from a different point of view: the variables are splitted into two groups such that all the variables belonging to the second are strictly functions of those of the first, so that they are completely irrelevant to the dynamics. Making the first group as small as possible corresponds to solving the original problem. While there are no precise recipes for proceeding along such a line, this procedure recently proved to be very powerful allowing to show that some infinite-dimensional set of equations admit finite-dimensional attractors [4].

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From the analysis sketched above, it is clear that the mathematically correct procedure that must be followed in the study of an infinite-dimensional system is to find a relatively low-dimensional manifold containing the asymptotic dynamics. In the present letter we want to show that such a way of reasoning, although correct, can lead to paradoxical results which in turn call for some revision of the method.

An indirect and *a posteriori* way of looking at the relevant degrees of freedom is via the fractal dimension D as computed from the Lyapunov exponents in terms of the Kaplan-Yorke formula [5]. The smallest integer $n > D$ obviously represents a lower bound to the number of active degrees of freedom.

Following this line of thought, we have studied the case of a class-B [6] bidirectional ring laser. For this class of lasers, experimentally investigated in ref. [7], the atomic polarization can be adiabatically eliminated, while one must take into account the spatial dependence of the population inversion $\Delta(x)$ along the axial resonator coordinate x. Hence the model is infinite dimensional. Usually $\Delta(x)$ is expanded in Fourier modes, Δ_n, and only the first two terms ($n = 0, 1$) are kept. In this letter we explore the range of validity of this truncation, showing that in many realistic conditions a finite but large number of modes has to be taken into account.

It is convenient to write the model equations in terms of suitably rescaled variables and parameters. In analogy with the single-mode case [8] we introduce the smallness parameter

$$\varepsilon \equiv \sqrt{\frac{\gamma}{\varkappa}}, \tag{1}$$

(γ and $\varkappa$ being the decay rates of population and field, respectively), as a measure of the deviation of the zeroth mode of population from its equilibrium position

$$\Delta_0 = 1 + \varepsilon w . \tag{2}$$

Moreover, the harmonics Δ_n ($n > 0$), the field intensities $I_i = E_i E_i^*$ ($i = 1, 2$), and the pump parameter A are rescaled by a factor $(1 + \delta^2)$, where δ represents the field atoms detuning (expressed in terms of the adimensional time variable $\tau = \sqrt{\gamma\varkappa}t$).

Finally, a rescaling with ε is extended to all the other Fourier modes, as follows:

$$\Delta_n = \varepsilon^n D_n , \qquad n \geq 1 . \tag{3}$$

No shift value has to be considered as in eq. (2), since the equilibrium state is $D_n = 0$. The power ε^n is an «ansatz» that will be justified in the last part of this letter.

The equation for the class-B ring laser [6] can then be written as

$$\begin{cases} \dot{E}_1 = (1 - i\delta)(E_1 w + E_2 D_1^*) , \\ \dot{E}_2 = (1 - i\delta)(E_2 w + E_1 D_1) , \\ \dot{w} = A - (I_1 + I_2) - \varepsilon[w(1 + I_1 + I_2) + E_1 E_2^* D_1 + E_1^* E_2 D_1^*] , \\ \dot{D}_1 = -E_1^* E_2 - \varepsilon[D_1(1 + I_1 + I_2) + E_1^* E_2 w + \varepsilon E_1 E_2^* D_2] , \\ \dot{D}_n = -D_{n-1} E_1^* E_2 - \varepsilon D_n(1 + I_1 + I_2) - \varepsilon^2 D_{n+1} E_1 E_2^* . \end{cases} \tag{4}$$

From eqs. (4) it is immediately seen that, in the limit case $\varepsilon = 0$, the dynamics factors out into a closed set of seven equations linking E_1^0, E_1^{0*}, E_2^0, E_2^{0*}, w^0, D_1^0, and D_1^{0*}, plus a cascade of forward coupled equations for D_n^0 (the superscript 0 means order zero in ε):

$$\begin{cases} \dot{E}_1^0 = (1 - i\delta)(E_1^0 w^0 + E_2^0 D_1^{0*}), \\ \dot{E}_2^0 = (1 - i\delta)(E_2^0 w^0 + E_1^0 D_1^0), \\ \dot{w}^0 = A - (I_1^0 + I_2^0), \\ \dot{D}_1^0 = -E_1^{0*} E_2^0, \\ \dot{D}_n^0 = -D_{n-1}^0 E_1^{0*} E_2^0. \end{cases} \tag{5}$$

As a result, the standard truncation [9] turns out to be rigorously correct for $\varepsilon = 0$. Equations (5) have some noticeable properties: they are invariant under time reversal combined with the involution $R(w^0, D_n^0) \equiv (-w^0, -D_n^0)$. Therefore, all orbits symmetric under R-transformation exhibit conservative features [10]. In fact, it is easily proved that the Lyapunov exponents form couples of opposite values as in symplectic dynamics. Extended numerical simulations have not lead to discover any evidence of simmetry breaking except for the fixed-point solutions. As a consequence, eqs. (5), which can be considered as a zeroth-order approximation of model (4), cannot reproduce the asymptotic dynamics, as ε introduces a qualitatively relevant change, that is a nonzero damping. However, in so far as «short» time evolution is considered, eqs. (5) provide the most natural approximation. Moreover, notice that, if δ too is set equal to 0, the quantity

$$H = w^{02} + 2D_1^0 D_1^{0*} + I_1^0 + I_2^0 - A \ln(|I_1^0 - I_2^0|) \tag{6}$$

is a constant of motion. It represents the generalization of the energy of a Toda oscillator, the only conserved quantity existing in the single-mode case (*i.e.* $I_2^0 = D_1^0 = 0$). Extended numerical simulations suggest the existence of a second conservation law, although we have not been able to find an analytic expression (1). Therefore, since two of the three phases of E_1^0, E_2^0 and D_1^0 can be factored out [11], model (5) can in principle be reduced to a closed set of three equations which at most can exhibit one positive Lyapunov exponent as it has been confirmed by our simulations. For $\delta \neq 0$ there are no longer conserved quantities and the available phase space is densely filled.

For $\varepsilon \neq 0$, the first seven equations no longer form a closed set and one has to deal with the complete system. However, since the evolution of the n-th mode depends on the $(n+1)$-th only through a second-order term, the latter one can be neglected in the limit $\varepsilon \ll 1$. The resulting equation is linear in D_n and presents an only forward coupling, so that it is possible to obtain the analytical expression of the associated Lyapunov exponent:

$$\lambda = -\varepsilon(1 + \langle I_1 + I_2 \rangle), \tag{7}$$

where $\langle ... \rangle$ indicates a time average. From the equation for w,

$$0 = A - \langle I_1 + I_2 \rangle + O(\varepsilon), \tag{8}$$

so that

$$\lambda = -\varepsilon(1 + A). \tag{9}$$

(1) An extended report on the numerical analysis will be published elsewhere.

From expression (9) it is easy to prove that the attractor dimension is finite. Kaplan-Yorke formula requires first to order the Lyapunov exponents from the largest to the smallest one and then to start summing them up until the minimum number n_c is found such that the partial sum is negative. Since, from eq. (9), all the characteristic exponents associated with the higher-order harmonics are equal and negative, it is straightforwardly seen that there must be a finite such n_c, for every fixed ε, although it can be very large if this control parameter is small (indeed in the limit $\varepsilon = 0$, n_c is infinite, as discussed at the end of this letter). The number of equations that must be retained can be estimated from the sole knowledge of the Lyapunov exponents of the first seven equations. In fact, this set of variables is coupled with all the others only via a second-order term in ε. Therefore, we can reasonably assume that in the limit of small ε, their Lyapunov exponents are not significantly altered by the higher modes. Under this hypothesis, the numerical integration of the first seven equations yields seven Lyapunov exponents that have to be added to an infinite sequence of exponents equal to $-\varepsilon(1+A)$. The Kaplan-Yorke relation then allows us to estimate the dimension. In particular, the smallness of the negative exponents implies that the dimension can be very high, indicating that many modes have to be taken into account.

To verify this general scheme, we have numerically determined the Lyapunov exponents of the reduced model, for various realistic values of pump and detuning, and fixed $\varepsilon = 0.01$. The resulting Lyapunov dimension (in the 7-d space) is plotted in fig. 1. In some cases, it gets very close to the space dimension, suggesting that other modes have to be considered. Then, a larger number of harmonics has been integrated, confirming that the Lyapunov exponents of the first seven equations are not changed significantly for ε values up to 0.05 and pump parameter A as big as 4-5. In the same parameter region the Lyapunov exponents of the higher-order harmonics satisfy very well relation (9).

Another numerical experiment was devised to prove the correctness of the scaling hypothesis for the harmonics (eq. (3)). This relation is of fundamental importance for all our

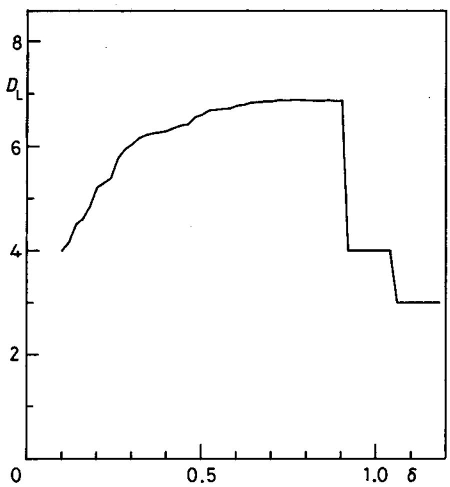

Fig. 1.

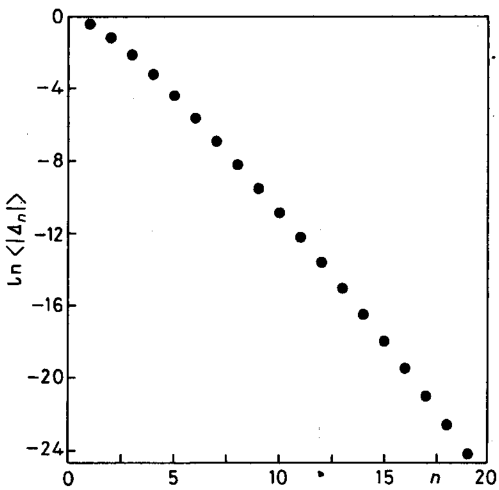

Fig. 2.

Fig. 1. – Lyapunov dimension of the reduced model (7 equations) for $\varepsilon = 0.01$, *vs.* δ. $A = 3.0$ in this case, but the results are practically independent of the value of this parameter, at least in the range we have tested ($0.5 \leq A \leq 4.0$).

Fig. 2. – Scaling behaviour of the average amplitude of the Fourier modes Δ_n. Parameter values: $A = 3.0$, $\varepsilon = 0.01$, $\delta = 0.6$.

results. Indeed we have neglected all ε^2 terms, under the implicit assumption that all D_n are of order 1 (*i.e.*, the amplitude of the harmonics scales with ε). To verify it we have integrated a high number of harmonics (twenty) and plotted the logarithm of the average of their amplitude *vs.* their order n (see fig. 2). As a matter of fact, the numerical results are in good agreement with the scaling law $d_n = (n+1)^2(k\varepsilon)^n$, where k is a constant of order 1 ($k \simeq 0.2$ in the case of fig. 2), that depends on the detuning and pump parameters.

An explanation of the leading part of such a scaling behaviour is found by again solving the model in the limit $\varepsilon = 0$. In this case, the last of eqs. (5) are perfectly integrable. The equation for the second harmonic, for example, is

$$\dot{D}_2^0 = -D_1^0 E_1^{0*} E_2^0 = +D_1^0 \dot{D}_1^0, \tag{10}$$

so that

$$D_2^0 = \frac{1}{2} D_1^{0^2} + a_2, \tag{11}$$

where a_2 is a constant of motion. The general solution is

$$D_n^0 = \frac{1}{n!} D_1^{0^n} + \sum_{k=0}^{n-2} \frac{1}{k!} a_{n-k} D_1^{0^k}, \tag{12}$$

where all a_n's are constants of motion. If they are chosen to be $O(1)$ at time zero, the D_n's are $O(1)$, at all times, in agreement with the initial ansatz. By switching on a small ε, the a_n's become slowly varying functions so that it is convenient to pass from D_n to a_n variables as the former ones exhibit a «rapidly» oscillating behaviour as well,

$$\dot{a}_n = -\varepsilon[(1 + I_1 + I_2)\, a_n + a_{n-1}(D_1(1 + I_1 + I_2) + wE_1^* E_2)]. \tag{13}$$

In the rescaled time units $\tau' = \varepsilon\tau$, eq. (13) is approximately equivalent to a Langevin equation with multiplicative and additive noise, where the stochastic forces are replaced by chaotic rapidly oscillating functions. By assuming $\langle |a_{n-1}| \rangle = 1$ by definition, it immediately follows that $\langle |a_n| \rangle = k' \simeq O(1)$, the constant k' being related to the k numerically found (we have to go back to D_n's). A quantitative estimate of k' is far from obvious as it depends on the correlations inevitably present in the chaotic motion. The initial ansatz is anyway definitely justified.

Finally, notice that the ε^2 terms so far neglected play anyhow a relevant role in providing a back-coupling from higher to lower modes. As a consequence, the dynamics of the original seven variables is really affected by such modes, although one must go to increasingly smaller length scales in order to observe this effect. Therefore, the limit $\varepsilon \to 0$ is a singular limit. Simultaneously, the amplitude of higher Fourier modes shrinks to zero, and the dimension diverges to infinity. This is because one has to reach an increasing observational resolution to find saturation in the dimension. In other words, the asymptotic manifold is so thin along many directions as to be practically indistinguishable from a sheet. Therefore, we suggest that the request for finding the dimensionality of the inertial manifold from the Kaplan-Yorke relation can, in some instances, be too strong a requisite, if one is not interested in reaching too large resolutions.

The concept of active degrees of freedom within preassigned length scales, may be a fruitful approach to describe nonlinear systems. In particular, in the case discussed in our letter, we might conclude that the standard truncation is correct up to scales ε^2 in the amplitude of the Fourier modes. However, we should also notice that a more realistic model for the ring laser should also take into account the diffusive motion of atoms in the active medium. This effect contributes to wash out the population inversion grid, not only rendering higher-order harmonics even smaller, but also associating them with faster decay rates. This last phenomenon, introducing stability coefficients independent of ε, eliminates the previous paradoxical result.

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