SHIL'NIKOV CHAOS: HOW TO CHARACTERIZE HOMOCLINIC AND HETEROCLINIC

BEHAVIOUR

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ABSTRACT

We introduce the concepts of Shil'nikov chaos and competing instabilities in a nonlinear dynamics including at least a saddle focus and a saddle point, a parameter change induces a smooth transition from a homoclinic to a heteroclinic trajectory.

In terms of the return map to a given Poincaré section, the two trajectories have the following characterization. In the homoclinic case, the global behavior is recovered from the local linear dynamics within a unit box around the saddle focus. The heteroclinic case requires the composition of two linearized maps around the two unstable points.

By an exponential transformation the geometrical map yields the return map of the orbital times. This new map represents the most appropriate indicator for experimental situations whenever a symbolic dynamics built on geometric position does not offer a sensitive test. Furthermore the time maps display a large sensitivity to noise. This offers a criterion to discriminate between a simulation (either analog or digital) with a few variables and experiment dealing with the physical variables embedded in the real world and thus acted upon by noise.

1. Theory of Shil'nikov homoclinic chaos

Shil'nikov dynamics corresponds to orbits asymptotic to an unstable saddle focus in at least a 3D space. Limiting to a 3D space let us call $\omega + \omega$ the pair of complex eigenvalues on the stable ($\chi < 0$) manifold and $\gamma > 0$ the eigenvalue in the unstable direction orthogonal to the plane.

Let us consider a dynamics where all fixed points are unstable, within a given range of control parameters. We call such situation a regime of competing instabilities. In physical implementations we can adjust the control parameter in order to isolate a non zero set of initial conditions such that all trajectories departing from there approach asymptotically the unstable saddle focus and remain at a finite distance from all other fixed points. In such a case, under the Shil'nikov
A single orbit of this type spiralling around an unstable saddle focus $S$ is qualitatively sketched in Fig. 1.

With the understanding that the only interesting dynamical features occur around point $S$ we obtain a global description by just studying the linearized dynamics within a small box around $S$ (Fig. 2).

In Fig. 2, we orient the three axes along the eigenvectors with $x$-y coinciding with the stable plane and $z$ being the expanding direction.

We take the $\mathbb{T}$ plane (vertical plane of equation $x=0$ containing a face of the cube) as the Poincaré section and we calculate the return map for the coordinate $z$. Starting at $t=0$ at $z=0$ on $x=1$ ($y$ is irrelevant for the following considerations) the phase point leaves the upper cube side $z=1$ at time $\mathcal{T}$ such that

$$1 = z_0 e^{\mathcal{T}}$$

from which it results

$$\mathcal{T} = -\log z_0$$

The horizontal coordinate $x$ evolves over the same time as

$$x(\mathcal{T}) = x_0 - \alpha \mathcal{T} \cos \omega \mathcal{T}$$

since the initial condition is $x(0)=1$. Neglecting a phase shift due to $y$ position, we constrain the motion external to the box to a rigid translation (see dashed trajectories).

$$x(\mathcal{T}) \rightarrow z_1$$

besides an offset $\xi$ added at each turn and which may be considered as a second control parameter, the first one being the ratio $|x_0/z_0|$.

Using relation (3) and writing $z$ as $z_{n+1}$ and $z_n$ as $z_n$ we obtain the return map

$$z_{n+1} = \frac{\alpha}{\mathcal{T}} \cos (\omega \log z_n) + \xi$$

which describes the homoclinic orbit.

The map (5), even though representing a sensible global description, may provide a poor experimental criterion whenever the $z$ coordinates on the $\mathbb{T}$ plane are clustered in a small region. A lack of experimental sensitivity appears in experimental return maps which do not display the nice features that Eq(5) provides for the theory. Such was the case for the Belousov-Zhabotinski reaction. On the other hand, the above behaviour appears rather universal whenever one can isolate a spiral type orbit, as it occurs in Lorenz or Rossler chaos.

In dealing with a quantum optical experiment, whose details will be reported in the next section, we introduced a more sensitive dynamical indicator $z$. Based on the logarithmic relation between position $z$ on the $\mathbb{T}$ plane and times $\mathcal{T}$ that the orbits take to return to that plane, and assuming that the relevant time is that spent in the box of Fig. 2, map(5) transforms via relation(3) into a return map for orbital times. We
rescale $\zeta$ as $T = \gamma \zeta = -\log z$ and obtain

$$T_{n+1} = - \ln [\exp(-\lambda/\gamma T_{n}) \cos(\omega/\gamma T_{n}) + i] = - \ln [\nu(T) + i],$$

(6)

Comparison of Eqs.(5) and (6) shows the enhanced sensitivity to fluctuations of the $T$ map with respect to the $z$ map. Indeed, suppose that the offset $e$ from homoclinicity is affected by a small amount of noise. The sensitivities of the two maps to such a noise are given, respectively, by $\psi_T/e = 1$ and

$$\partial T/\partial T = [\nu(T) + i]^{-1}.$$  

(7)

This sensitivity factor acts as a lever arm whenever $\psi_T = e$ becomes very small. Note the following: (1) This is not deterministic chaos; in fact, large fluctuations can be expected even for a regular dynamics, implying a fixed point $T^*$, (2) It is not associated with the homoclinicity condition $e = 0$; in fact, for finite $e$ there may be a $T^*$ such that $\psi_T = e = 0$.

Since a homoclinic orbit is the dynamic counterpart of repeated decays out of an unstable state, the result is like repositioning the initial condition in an experiment on a single decay. Here the repetition is automatically provided by the contracting motion asymptotic to the stable manifold. As a consequence, superposed upon the deterministic dynamics (either regular or chaotic), the high sensitivity (Eq.(7)) may provide a broadening of the $T$ maps not detectable in the $z$ maps whenever noise in the offset $e$ is present.

In fact, the model description $x = F(x)$ of a large system in terms of a low-dimensional dynamic variable $x$ is just an ensemble-averaged description, and residual fluctuations on position $x$ must be considered at some initial time, even though the successive evolution is accounted for by a deterministic law. In our case such a fluctuation is a stochastic spread $\delta x$ on the offset $e$ of the position $z$.

As shown in Fig. 3, the same amount of $\delta x$ in Eqs.(5) and (6) leaves the $T$ maps unaltered, while it strongly affects the $T$ maps.

If we specialize the map parameters $a, b, w,$ and $e$ to a regular orbit (fixed points both in $z$ and $T$ spaces), introduction of $\delta e$ does not broaden the $z$ point, while the $T$ point broadens.

For example, the values $a/b = 0.98, w/b = 2.98,$ and $e = 0.01$ yield one fixed point $T^* = 5.327$, with a sensitivity $\delta T^*/\delta e = 182$ (Fig. 3(c)).

Note that the noise effect reported here has nothing to do with additive noise effects on return maps already described. Indeed, the latter effects refer to the scaling behavior near stationary bifurcations, whereas our data refer to transient fluctuation enhancement, and they do not leave a permanent mark (such as an orbital shift or broadening).

Thus, while Shil'nikov chaos is a deterministic effect described on

Fig. 3. Numerical iteration maps for Shil'nikov chaos. Parameter values: $a/b = 13.0, w/b = 0.986, e = 0.01$. (a) and (b), $T$ maps without and with noise $\delta e = 10^{-4}$, respectively. (c) Stable fixed point of the regular dynamics, broadened by a noise $\delta e = 10^{-4}$. 

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average by the backbone of the $z$ or $T$ maps, the superposed thickening is a noise effect peculiar to $T$ maps and undetectable in $z$ maps. This new effect is a specific indicator of intrinsic fluctuations, and it permits a demarcation line to be drawn between a real-life experiment and a model simulation, from which this second feature is absent.

In order to explore the regular behavior of these closed orbits, we take the fixed point of map(6)

$$
T^* = -\ln\left[\exp\left(-\lambda / \gamma T^* \cos\left(\omega / \gamma T^*\right)\right) + 1\right]
$$

(8)

Eq. (8) gives a stable fixed point, provided Shil'nikov condition (i) is violated, that is.

Solving transcendental Eq. (8) and plotting the Poincaré frequency $1/T$ versus the control parameter $\xi$, yields two different items, namely:

1) a staircase region implying hysteresis cycles
2) a logarithmic divergence for small $\xi$.

2. Experimental implementation of Shil'nikov dynamics

For pedagogical reasons we have collected in Sec. 1 the main features of Shil'nikov chaos as well as those of a regular (non-chaotic) dynamics and here we describe the corresponding experiments. As a matter of fact, things have gone in the reverse order: we first found evidence of spiral type orbits, including large time fluctuations, or regular periodic cycles with control parameter as in (i) of Sec. 1; then we looked in the theoretical literature and found that, using the orbital period as a dynamical indicator more sensitive than Poincaré position, we could nicely describe what was previously treated only at a qualitative level, in terms of a symbolic dynamics coding the number of spirals around the saddle focus.

Our experimental setup consists of a single mode CO$_2$ laser with an intrinsically electro-optic modulator. A signal proportional to the laser output intensity is sent back to the electro-optic modulator. Single mode CO$_2$ lasers have a dynamic behavior described by two coupled differential equations, one for the field amplitude and the other for the population inversion, the fast polarization being adiabatically eliminated from the complete set of Maxwell-Bloch equations. Thus, the presence of feedback introduces a third degree of freedom. When the feedback loop is so fast that it produces a practically instantly adapted loss coefficient, it does not modify the phase-space topology. On the other hand, if the time scale of the feedback loop is of the same order as of the other two relevant variables, the system becomes three dimensional. With suitable normalizations such a system is described by three first-order differential equations for the laser intensity $x(t)$, the population inversion $y(t)$, and the modulation voltage $z(t)$ as follows:

$$
\begin{align*}
x &= -K_x \left[1 + \frac{\gamma}{\gamma^*} \sin^2(z) - y\right], \\
y &= -\frac{\gamma_z}{\gamma}(y + xy - A), \\
z &= -\beta (z - B + r x),
\end{align*}
$$

(9)

where $K_x = (c/l) T$ is the unmodulated cavity-loss parameter, $L$ is the cavity length, $T$ is the effective transmission of the cavity, and $\gamma_z$ is the population decay rate. The intensity $x(t)$ is normalized to the saturation intensity; the population inversion $y(t)$ is normalized to the threshold inversion $z(t)$ is the modulation voltage normalized to $\pi/\sqrt{2}$ with $\pi/\sqrt{2}$ the $\pi/2$ modulator voltage; $A$ is the normalized pump parameter; $\beta$ is the damping rate of the feedback loop; $r$ is a coupling coefficient between the detected intensity $x(t)$ and the normalized $z(t)$ voltage; $B$ is the bias voltage applied to the electro-optic modulator; and $\gamma_z = (1 - T)/T$.

In Fig. 4 we present a schematic view of the trajectory in the three-dimensional space, obtained by a linear stability analysis of the motion around the stationary points, and qualitative connections between the linear manifolds (dashed lines).

![Fig. 4. Schematic view of a trajectory in the phase space when the dynamics are affected by all three unstable stationary points.](image)

From an experimental point of view we are able to visualize $(x-z)$ phase-space projections, obtained by feeding onto a scope the photodetector signal proportional to the laser output intensity $x(t)$ and the feedback voltage $z(t)$. This phase-space projection consists of closed orbits visiting successively the neighborhoods of the three unstable stationary points 0.1, and 2.
The local chaos around point 1, established at the end of a subharmonic sequence, has been characterized by standard methods as power spectra and correlation dimension measurements.

The competition of the three instabilities in controlling the global features of the motion was described in Ref. 3. There $\omega / \omega'$ was adjusted major then one showing regular behavior and experimental evidence of items i) and ii) of sec. 1. Here we adjust the control parameters in order to have a dominance of the saddle focus, so that the motion consists of a quasi-homoclinic orbit asymptotic to it.

In fig. 5 we report experimental plots of the laser intensity vs. time for two slightly different conditions. Keeping a fixed population inversion (pump 1.5 times above the threshold value) the system has two control parameters, the bias $B$ and the gain $G$ of the amplifier driving the electro-optic modulator.

We have kept $B$ fixed at $B=0.35$ and increased $r$ from $r=0.467$ in fig. 5a) to $r=0.491$ in fig. 5b). Figure 5b) shows clear evidence of a homoclinic orbit in the two long transients, which produce a lengthy permanence in a phase space region of almost constant intensity. This appears more clearly in the corresponding phase space projections (fig. 5a) and b)).

For comparison we give in fig. 5c) a photographic exposure (over 1 s) of 30,000 orbits as that of fig. 5a), to show the stability of shape. We see that the first three large oscillations of fig. 5 have strong anharmonic distortions and display common features over different repetitions, while the small oscillations around the saddle focus display slight differences from pulse to pulse. These latter oscillations are ruled by the linearized dynamics which consists of a contracting spiral $-\omega X + t \cos(\omega t)$ on the stable manifold and of an expansion $\exp \delta t$ along the unstable direction.

Before we discuss comparison with Shil'nikov theory, a crucial question arises: how much of the spread in the return times has to be attributed to point 2 or 0? Indeed, we have quasi-heteroclinic orbits visiting the surroundings of the two unstable points 2 and 0. But in our experimental situation the dynamics can be assimilated to a quasi-homoclinic orbit around point 2, which is thus mainly responsible for the spread in return times. This is easily proved by measuring the spread $T$ in the residence times $T_n$ around 0 (zero intensity stripes) and the spread $T_2$ in the residence times $T_n$ around 2 (complementary stripes, such that $T_0 + T_2$ is the total orbital time). In fig. 5, which shows typical time sequences used to build the two averaged relative spreads are approximately

$$\frac{\langle \Delta T_0 \rangle / T_0}{\langle \Delta T_2 \rangle / T_2} \sim 14\%, \quad \frac{\langle \Delta T_0 \rangle / T_0}{\langle \Delta T_2 \rangle / T_2} \sim 80\%,$$

$$\frac{\langle \Delta T_0 \rangle / T_0}{\langle \Delta T_2 \rangle / T_2} \sim 40\%, \quad \frac{\langle \Delta T_0 \rangle / T_0}{\langle \Delta T_2 \rangle / T_2} \sim 250\%.$$

The comparison shows that point 0 introduces a perturbation around 14% with respect to pure homoclinicity, that is, the orbital regularity is ruled mainly by point 2. Thus a theoretical approach to our experiment in terms of homoclinic chaos appears justified.

We measure the time spacings by setting a threshold circuit near the top of the largest peak of the intensity signal. A time-to-amplitude converter (TAC) yields the sequence $T_1$ of successive time spacings,
which is then classified as a statistical distribution by a multichannel pulse height analyser, or stored in a digitizer, so that correlation functions or iteration maps can be sorted out.

The statistical distribution of return times is a broad featureless curve which does not offer cues on the ordering of $T_r$. On the contrary, the iteration map ($T_{ls}$ vs. $T_r$) displays a regular structure (fig. 7a). To check whether we are in the presence of a one-dimensional (1D) iteration map, and the remaining thickness is due to the observation technique, or the map is more that 1D, we report in fig. 7b) the iteration maps corresponding to three regular situations.

![Iteration Maps](image)

**Fig. 7.** Experimental iteration maps of the return times. a) refers to $r=0.487$ and to $B=0.350$. b) shows the maps corresponding to regular periodic situations, namely, 1) an electronic oscillator, 2) the laser in a regular periodic regime and 3) the laser just at the onset of the instability but still with a regular period.

In the absence of fluctuations in $T_r$ they should be pointlike (the image of a stable fixed point). In fact 1) corresponds to an electronic oscillator and it just shows the resolution of the TAC, 2) corresponds to the laser in a regular periodic regime away from the Shil'nikov instability, and 3) corresponds to the laser on the verge of the instability but still with a regular period. In this last case, the fluctuation associated with the nearby transition shows that, even without chaos in the return time, the close approach to an instability point introduces a fluctuation enhancement, which has no theoretical counterpart in the current treatment of deterministic chaos. To deal with this broadening, the dynamical equations should include a statistical spread in the injection coordinate at the Poincaré section near the saddle focus, to account for the macroscopic character of the experimental system. As it was shown in ref. 10, even though this spread has no relevance on the average dynamics, it contributes a large transient fluctuation whenever the system decays from an unstable point.

3. Toward heteroclinicity: role of points 0 and 1

We have already seen that we can isolate quasi-homoclinic conditions when most of the time fluctuations are due to the residence time around point 2.

An intensity plot seems to approach dangerously the zero-intensity state, however this does not change yet the homoclinic characterization. More frequently, we have situation where 0 and 1 are very close and the decaying spiral around 0 is followed by a weakly amplifying spiral around 1. Evidence of this is offered by fig. 6b and 5a.

To better emphasize this point we report a new experimental plot (fig. 8a) and a corresponding numerical experiment (fig. 8b).

![Intensity Plots](image)

**Fig. 8.** a) Experimental time evolution of the laser intensity for $B=0.433$ $V=0.558$ b) Time evolution in numerical simulation. Parameter values for a 3 level model $K=2.0 \times 10^{-5}$; $\xi=0.10^2$; $\eta=4.10^6$; $B=3.10^6$; $A=3.95$; $r=.2$; $B=1.86$; $r=.603$

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A 3 level model for CO₂ molecular introduced by Shinizu et al. was used by the two groups working on L.S.A. (Ref. 11, 12 plus private communication). In our case, we have tested both 2 and 3 level models but in our time range we have not found relevant differences. This may be due to the fact that the feedback dynamics goes over time of the order of 10-100 us and this time range is not sensitive to the further decays introduced by the 3 level model.

4. Conclusions

We conclude stressing the universality of the phenomena here reported. The spiraling around foci and associated phenomena should occur in all Lorenz type or Roessler type dynamics. Evidence of it is in the recent report on Lorenz chaos as well as in the above reports on B-Z chemical instabilities.

The best assessments are obtained by time maps as said above. Time maps are now being used also by other investigators.

The technique of a local linear dynamics to get a global description can be extended to the heteroclinic case, in terms of a composition of two linearized maps around the two unstable points. A preliminary investigation of this kind has lead us to a conjecture on stabilization of orbital times in heteroclinic dynamics which seems relevant for biological clocks. This subject is developed elsewhere.

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