Transient statistics in a CO_2 laser with a slowly swept pump

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The near-threshold behavior of a single-mode CO_2 laser has been investigated by applying a linear modulation to the excitation current. The associated delayed bifurcation has been characterized in terms of first-passage-time distributions for different sweep rates. A theoretical model provides good agreement with the experimental data. In particular, we are able to assign the initial photon number below threshold.

I. INTRODUCTION

In this paper we report the behavior of a single-mode homogeneously broadened CO_2 laser when the gain parameter is swept from a negative value (laser below threshold) to a positive one (laser above threshold), so that the dynamical system goes through a bifurcation point. The modulation of the control parameter (gain) is done by repeatedly applying to the discharge current a linear ramp, followed by a plateau. As the gain does not assume at any moment the stationary value associated with the instantaneous current value, the observed bifurcation is delayed with respect to the time at which the control parameter passes through the instability. The delayed bifurcation was theoretically treated by P. Mandel *et al.*,[1] the statistical features of transient behavior have been studied either by photon counting distributions[2] or passage time techniques.[3] In the experiment reported here, we have a combination of delay and transient statistics.

The influence of additive white noise on delayed bifurcations in a single-mode class-*A* (Ref. 4) laser was studied by Broggi *et al.*[5] by using a Fokker-Planck formalism. Van den Broeck *et al.*[6] and recently Zeghlache *et al.*[7] have analyzed this problem by a direct analysis of the moment equations for the field amplitude, again in the case of a class-*A* laser. The validity of these approaches was confirmed on analog simulations by Mannella *et al.*[8] and by Stocks *et al.*[9]

As for the experimental technique of a swept pump parameter, preliminary results on transient bistability were reported by Glorieux *et al.* and by Arimondo *et al.*[10] in CO_2 lasers with a saturable absorber. In that case however the dynamical behavior was due to the joint interaction of the laser medium and the saturable absorber. Here we want to characterize the statistical features of the pure laser under a pump sweep. The same behavior was observed by Scharpf *et al.*[11] at the threshold of an Ar^+ laser by varying the cavity losses, and by Mecozzi *et al.*[12] in a semiconductor laser by switching the biasing current. While Ref. 11 does not deal with statistical features, Ref. 12 indeed tackles our same problem. Having scaled however the problem from semiconductor to a CO_2 laser allows to work on a slower time scale and hence to gather more accurate data, which permit a quantitative comparison between experiment and theory.

An operational definition of this dynamical bifurcation point (laser switch-on), delayed with respect to the static one (gain equal to zero), can be given in terms of a "first passage time", that is, the time spent by the laser in reaching for the first time an intensity value significantly different from noise. The fluctuations due to spontaneous emission make the first passage time a stochastic variable, and consequently a stoichastic treatment of the problem is needed. A general theory for a class *A* laser was provided by Torrent *et al.*[13]

For class-*B* lasers, ruled by two coupled equations for field and population, delayed bifurcation phenomena present features not considered in the above theories.[1, 5–7, 13] In fact, they are sufficient in the linear regime provided the population is kept constant, that is, they describe only loss modulation. A deterministic treatment for class-*B* laser with slowly swept loss parameter was recently done by Erneux *et al.*[14]

In the case of pump modulation one has to further consider the delayed action of the varying population on the time dependent gain parameter. For this reason we present here below a specific stochastic treatment which is apt for pump modulation. In Sec. II we report the relevant features of the measuring set up. Sec. III presents a theory of the delayed bifurcation adapted to a class-*B* laser. In Sec. IV we compare the experimental data with the theoretical expectations of Sec. III. In the conclusion we emphasize the relevance of a modified model to deal with class-*B* lasers.

II. EXPERIMENTAL SETUP AND RESULTS

The experimental setup is shown in Fig. 1. It consists of a CO_2 laser tube, with Brewster angle windows, placed inside a resonator 1.5-m long. One of the reflectors of the laser cavity is a grating mounted in autocollimation configuration, in order to select the $P(20)$ line at 10.6 μm. The other is a partially reflecting Ge mirror ($R = 95\%$) with a 3-m radius of curvature. The coupling mirror is mounted on a hollow cylindrical piezoelectric translator (PZT) in order to control the detuning between the center of the molecular line and the frequency of the

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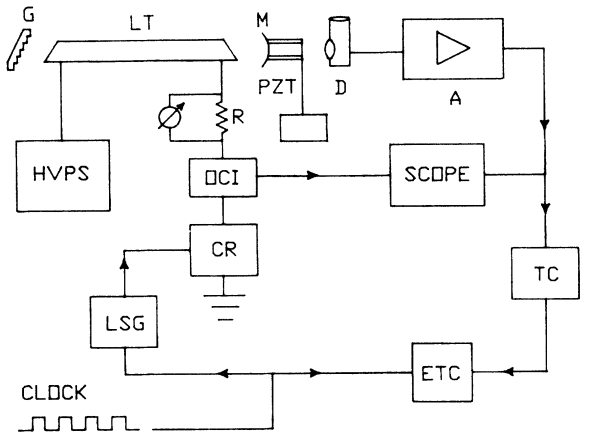

FIG. 1. Experimental setup. LT, laser tube; G, grating; M, coupling mirror mounted on a piezo translator; HVPS, high voltage power supply; OCI, optically coupled isolator; CR, current regulator; LSG, linear sweep generator; ETC, electronic time counter; TC, trigger circuit; D, Hg-Cd-Te detector; A, low-noise amplifier; R, precision resistor.

cavity mode. The inflow CO_2 laser is pumped by means of a dc discharge. The power supply is current stabilized to better than 0.05%. The discharge length is 0.8 m and the gas mixture is composed of CO_2 (15.4%), H_2 (2%), N_2 (14.2%), and He (68.4%) at a total pressure of 20 Torr, measured at the gas inlet of the laser tube. The electronic circuit for the current regulation is externally modulated by a linear ramp followed by a plateau. The mean value of the discharge current is measured on a precision resistor R. The current is directly monitored by means of an optically coupled isolator in series with the regulator. The laser output intensity is detected by a liquid-N_2-cooled Hg-Cd-Te detector with a rise time $\tau \leq 10$ ns. The photodetector signal is amplified and sent together with the current signal to a digital oscilloscope to monitor the temporal evolution.

An electronic time counter measures the time interval between the start of the linear ramp and the laser switch-on spike. The counter repeats a sequence of 1000 measurements and displays the mean value and the standard deviation of the first-passage-time distributions.

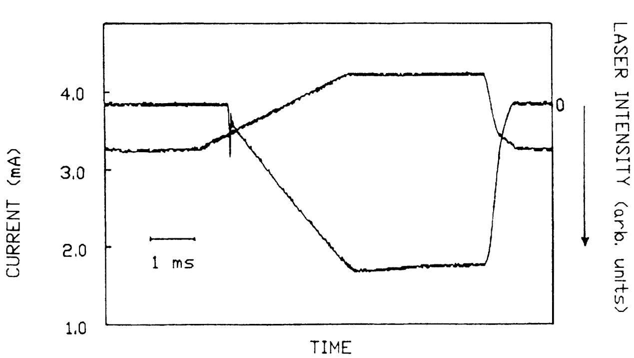

FIG. 2. Time evolution of both laser intensity and excitation current which displays the linear ramp and the two plateaux.

TABLE I. Mean value ($\langle t^* \rangle$) and standard deviation (δt) of first-passage-time distributions as a function of the slope ϑ of the linear ramp.

ϑ (A/s)	$\langle t^* \rangle$ (μs)	δt (μs)
0.323	1271	16.3
0.470	943	11.4
0.589	809	9.4
0.738	700	8.1
0.812	644	7.4
0.959	581	6.8
1.070	541	6.3

In Fig. 2 we report the time evolution of the laser intensity and the corresponding current modulation. Experimental evidence of the delayed bifurcation at the threshold of a CO_2 laser (also reported in Ref. 15) is clearly shown here, as the laser switch-on and switch-off take place for two different values of the discharge current.

By setting the zero of the time scale at the start of the linear sweep (delayed by 100 μs with respect to the TTL clock signal due to the bandwidth of the electronic system), we define t^* as the time at which the laser switches on. In Table I we report $\langle t^* \rangle$ and the associated standard deviations δt as a function of the slope ϑ of the linear ramp.

III. THEORETICAL ANALYSIS

A. Deterministic treatment

Our results can be explained in terms of the class-B laser model which consists of two rate equations describing the photon number n and the population inversion Δ between the two resonant levels as follows:

$$\dot{n} = 2\left[-kn + \frac{Gn\Delta}{2} \right] , \quad (1)$$

$$\dot{\Delta} = -\gamma_{\parallel}(\Delta - \Delta_0) - 2Gn\Delta . \quad (2)$$

k and $\gamma_{\parallel}$ are the decay rates of field and population inversion, respectively, Δ_0 is the population inversion provided by the pump mechanism, and $G = 2c\lambda^2/8\pi V \tau_{sp}\gamma_{\perp} = 5.3\times10^{-8}$ s^{-1} is the field-matter coupling constant [$V = 34.4$ cm^3 is the mode volume, $\gamma_{\perp} = \pi\Delta\nu_c = 4.4\times10^8$ s^{-1} is proportional to the collisional broadening, and $\tau_{sp} = 3.33$ s is the spontaneous radiative lifetime of the $P(20)$ transition]. Since the excitation pump current is linearly modulated, we assume Δ_0 as a linear function of time, that is,

$$\Delta_0 = \Delta_0(t) = B_0 + \beta t .$$

In the linear regime, where we limit our observations, the photon number n is smaller than the saturation value $n_s = \gamma_{\parallel}/2G = 4.7\times10^{10}$ so that the last term in Eq. (2) can be neglected. In this case the solution, with the initial condition

$$\Delta(t=0)=\Delta_0=B_0 ,$$

is given by

$$\Delta(t)=B_0+\beta t-\frac{\beta}{\gamma_\parallel}(1-e^{-\gamma_\parallel t}) .$$

Equation (1) can be now rewritten as

$$\dot{n}=2n\Gamma(t) , \tag{3}$$

where $\Gamma(t)$ is the total photon gain

$$\Gamma(t)=a_0+\alpha t-\frac{\alpha}{\gamma_\parallel}(1-e^{-\gamma_\parallel t}) \tag{4}$$

with $A_0=\Gamma(t=0)=-k+(G/2)B_0<0$ and $\alpha=(G/2)\beta$. a_0 represents the total initial gain and it has obviously a negative value (laser below threshold), while α is the sweeping rate.

The solution of Eq. (3) is

$$n(t)=n_0\exp\left[2\int_0^t\Gamma(\tau)d\tau\right] . \tag{5}$$

We define $\bar{t}$ as the time at which $\Gamma(\bar{t})=0$ and t^* the time at which, during the exponential amplification, the photon number reaches a certain threshold value below the saturation one. From Eq. (5) we have $t^*>\bar{t}$ because for any $t<\bar{t}$

$$\int_0^t\Gamma(\tau)d\tau<0 ,$$

so that the delay at the bifurcation is the time necessary to overcome the stability accumulated in the interval $[0,\bar{t}]$. Solving Eq. (5) we obtain

$$\ln\left[\frac{n}{n_0}\right]=2\left[a_0-\frac{\alpha}{\gamma_\parallel}\right]t+\alpha t^2+\frac{2\alpha}{\gamma_\parallel^2}(1-e^{-\gamma_\parallel t}) . \tag{6}$$

Assuming $\gamma_\parallel\simeq 5\times 10^3\ \mathrm{s}^{-1}$ (Ref. 16) and $t>500\ \mu$s, the term $e^{-\gamma_\parallel t}$ is less than 0.08 and it can be neglected with respect to unity. Finally, inverting Eq. (6) we have

$$t=\left[\frac{1}{\gamma_\parallel}-\frac{a_0}{\alpha}\right]\pm\left[\left[\frac{1}{\gamma_\parallel}-\frac{a_0}{\alpha}\right]^2-\frac{2}{\gamma_\parallel^2}+\frac{1}{\alpha}\ln\left[\frac{n}{n_0}\right]\right]^{1/2} . \tag{7}$$

This relation expresses the time t at which there are n photons in the laser cavity as a function of the initial conditions n_0 and a_0 and of the sweeping rate α.

B. Stochastic treatment

A more accurate description of the laser threshold instability requires the addition of a stochastic term in the field equation which takes into account the spontaneous emission fluctuations:

$$\dot{E}=-kE+\frac{G}{2}E\Delta+\xi(t) . \tag{8}$$

The function $\xi(t)$ represents a Gaussian white noise with zero mean value and δ-correlated

$$\langle\xi(t)\rangle=0 ,\qquad \langle\xi(t),\xi(t')\rangle=\varepsilon\delta(t-t') . \tag{9}$$

The field solution is given by

$$E=h(t)\exp\left[\int_0^t\Gamma(\tau)d\tau\right] ,$$

where

$$h(t)=E_0+\int_0^t\xi(t')\exp\left[-\int_0^{t'}\Gamma(\tau)d\tau\right]dt'$$

$$\Gamma(t)=-k+\frac{G}{2}\Delta(t) .$$

Since in our notation the variable E is dimensionless, $h^2(t)$ represents the noise photon number. Assuming that the below-threshold field amplitude $E(0)$ has a Gaussian distribution

$$P(h)=\frac{1}{\pi\langle h^2\rangle}\exp\left[-\frac{h^2}{\langle h^2\rangle}\right] \tag{10}$$

the variance of $h^2(t)=n_0(t)$ is[13]

$$\langle h^2(t)\rangle=\langle n_0(t)\rangle=\frac{\varepsilon}{|a_0|}+\varepsilon\int_0^t\exp\left[-2\int_0^{t'}\Gamma(\tau)d\tau\right]dt' \tag{11}$$

where the first term represents the initial noise and the second the noise contribution along the interval $[0,t]$.

The photon number equation [corresponding to Eq. (5)] is

$$n(t)=n_0(t)\exp\left[2\int_0^t\Gamma(\tau)d\tau\right] . \tag{12}$$

This equation shows that at time t, the laser intensity is proportional to the noise $n_0(t)$ through an exponental amplification coefficient.

When the laser intensity reaches the threshold value we have

$$n(t)/n_0(t)=\exp\left[2\int_0^t\Gamma(\tau)d\tau\right]\gg 1$$

and we can approximate

$$h(t)\simeq h(\infty),\quad n_0(t)\simeq n_0(\infty) ,$$

$$\langle h^2(t)\rangle\simeq\langle h^2(\infty)\rangle,\quad \langle n_0(t)\rangle\simeq\langle n_0(\infty)\rangle ,$$

so that the solution of Eq. (12) is given by Eq. (7) with n_0 replaced by $n_0(\infty)$ [from now for simplicity we consider $n_0(\infty)=n_0$ and $\langle n_0(\infty)\rangle=\langle n_0\rangle$].

In order to obtain the moments of the passage time distribution we introduce the generating function

$$W(\lambda)=\langle e^{-\lambda t}\rangle=\int_0^\infty e^{-\lambda t}P(t)dt , \tag{13}$$

where $P(t)$ is defined as

$$P(t)=\int_0^t\delta(t-t(n_0))P(n_0)dn_0$$

and, from (10),

$$P(n_0)=\frac{1}{\pi\langle n_0\rangle}\exp\left[-\frac{n_0}{\langle n_0\rangle}\right].$$

In this way, the probability to observe a certain passage time t (corresponding to n photons) depends on the probability that the noise amplitude is n_0.

Finally, taking into account that

$$W(\lambda)=\int_0^n \exp[-\lambda t(n_0)]P(n_0)dn_0 , \tag{13'}$$

$$\langle t\rangle=-\frac{d}{d\lambda}\ln[W(\lambda)]|_{\lambda=0} , \tag{14}$$

$$\langle \Delta t^2\rangle=\left[\frac{d}{d\lambda}\right]^2\ln[W(\lambda)]|_{\lambda=0} , \tag{15}$$

and with the approximations

$$n\gg n_0,\quad \int_0^n P(n_0)dn_0\simeq 1 ,$$

we obtain

$$\langle t\rangle=\left[\frac{1}{\gamma_{\parallel}}-\frac{a_0}{\alpha}\right]\pm\left[\left[\frac{1}{\gamma_{\parallel}}-\frac{a_0}{\alpha}\right]^2-\frac{2}{\gamma_{\parallel}^2}+\frac{1}{\alpha}\ln\left[\frac{n}{\langle n_0\rangle}\right]-\frac{\psi(1)}{\alpha}-\langle\Delta t^2\rangle\right]^{1/2} , \tag{14'}$$

where $\psi(1)\simeq -0.577$ is the digamma function.[17] The last two terms in the square root are the stochastic corrections to the deterministic expression given in Eq. (7).

IV. INTERPRETATION OF DATA

As shown above the expression for the mean time $\langle t\rangle$ involves $\langle n_0\rangle$ which, in turn, is a complicated function depending not only on a_0 but also on α. If we set the initial value of the gain a_0 so close to threshold that the noise along the interval $[0,t^*]$ is negligible with respect to the initial noise, $\langle n_0\rangle$ can be considered independent of α. Moreover, as the population inversion $\Delta_0(t)$ and the discharge current $I(t)$ are proportional for small modulation amplitudes, it is convenient to rewrite

$$I(t)=\rho\Delta_0(t)=\rho B_0+\rho\beta t=I_0+\vartheta t ,$$

where $I_0=\rho B_0$ ($I_0=3.23\pm0.03$ ma) is the current corresponding to the lower plateau and $\vartheta=\rho\beta$ is the slope of the linear ramp. These two parameters are measured by using the response curve of the optically coupled isolator. If we now introduce the parameter $L=G/2\rho$, α and a_0 can be expressed in terms of the measurable quantities I_0 and ϑ as follows:

$$\alpha=L\vartheta,\quad a_0=-k+LI_0 .$$

Assuming $\gamma_{\parallel}\simeq 5\times10^3\ s^{-1}$ and $k=5\times10^6\ s^{-1}$ (which corresponds to a total transmission coefficient of 10%), we are able to fit the unknown constants of Eq. (14') (with plus sign) obtaining the following values, L in $(A\ s)^{-1}$ and a_0 in s^{-1}:

$$L=1.483\times10^9 ,$$
$$a_0=-2.11\times10^5 , \tag{16}$$
$$\ln\left[\frac{n}{\langle n_0\rangle}\right]-\psi(1)=18.7 .$$

The agreement between the experimental points and the fitted curve, shown in Fig. (3), is within 2%. The error bars associated with the experimental points, evaluated by the corresponding standard deviations, are negligible.

An accurate calibration of the laser power output determines the value of the photon number n at which we have performed the measurements (the threshold for the trigger circuit was settled at 50 mV which corresponds to a laser power output of 3.16 mW). As the mirror transmission is 5% the photon number is given by $n=3.16\ mW/[\hbar\omega(c/l)0.05]=1.65\times10^{10}$. This implies [from (16)] a mean photon number below threshold $\langle n_0\rangle\simeq220$, in agreement with previous results on a CO_2 laser with fast loss switching.[18]

As $G=5.3\times10^{-8}\ s^{-1}$ the factor $(\rho V)^{-1}$ relating the population inversion density Δ_0/V to the discharge current is

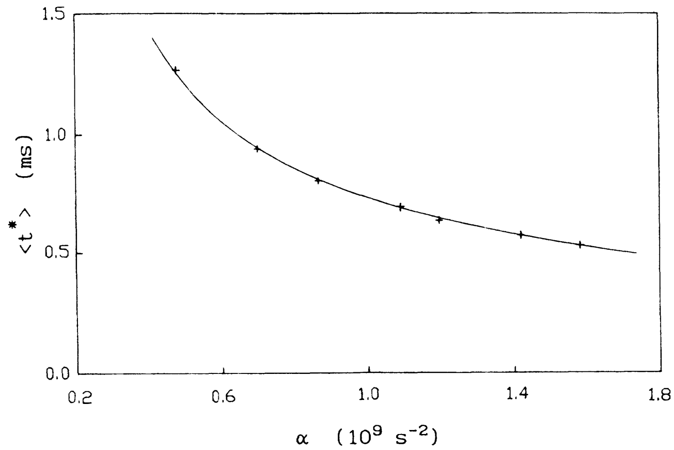

FIG. 3. Switch-on times $\langle t^*\rangle$ vs the sweep parameter α. The crosses denote the experimental points; the solid line represents the theoretical best fit with the present theory.

TABLE II. Delay at the bifurcation ($\langle t^*\rangle-\bar{t}$) as a function of the sweeping rate α.

$\alpha(10^9\ s^{-2})$	$\langle t^*\rangle-\bar{t}$ (μs)
0.479	639
0.697	458
0.873	392
1.094	340
1.204	306
1.422	276
1.587	256

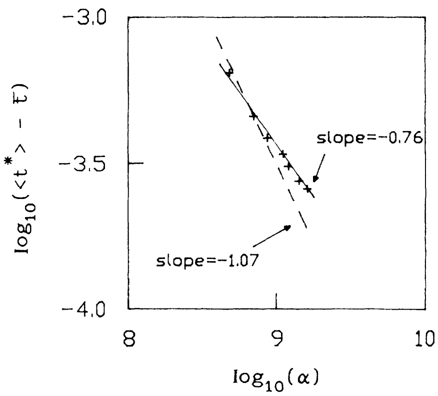

FIG. 4. Delay time $\langle t^* \rangle - \bar{t}$ vs the sweep parameter α in log-log scale. The crosses denote experimental points, the solid line denotes present theory, and the dashed line denotes class-A theory.

$$\frac{1}{\rho V} = 1.63 \times 10^{12}\ (\mathrm{mA\,cm^3})^{-1}\ .$$

This value agrees with that obtained by studying fast cavity loss modulation.[19]

By using the values of the other two fitted constants a_0 and L, we are able to determine the time $\bar{t}$ at which the total gain $\Gamma(t)$, given by Eq. (4), is equal to zero. In Table II the bifurcation delay time $\langle t^* \rangle - \bar{t}$ is reported as a function of the parameter α. In Fig. 4 the same relation is shown in a log-log scale which displays a power-law behavior as

$$(\langle t^* \rangle - \bar{t}) \propto \alpha^{-0.76}\ .$$

The corresponding class-A laser behavior (Ref. 13) gives a slope -1.07 (dashed line).

V. CONCLUSIONS

The near-threshold behavior of a CO_2 laser with slowly swept pump has been investigated by means of first passage time statistics. At variance with loss modulation, pump modulation is a sensitive test of the time scale of the molecular decays, so that the class-B laser model is needed. The linearization of Eq. (2) and the assumption of a linear modulation of the pump leads to an exact solution of the differential equation for the laser intensity, which implies the observed delay at the bifurcation. In the stochastic description the solution for the mean first passage time $\langle t \rangle$ is in good agreement with the experimental data and permits an evaluation of the laser mean photon number below threshold.

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[1]P. Mandel and T. Erneux, Phys. Rev. Lett. **53**, 1818 (1984); P. Mandel, Opt. Commun. **64**, 549 (1987).

[2]F. T. Arecchi, V. Degiorgio, and B. Querzola, Phys. Rev. Lett. **19**, 1168 (1967); F. T. Arecchi and V. Degiorgio, Phys. Rev. A **3**, 1108 (1971).

[3]F. T. Arecchi and A. Politi, Phys. Rev. Lett. **45**, 1219 (1980); F. T. Arecchi, A. Politi, and L. Ulivi, Nuovo Cimento **71B**, 119 (1982).

[4]We recall that the terms class A, B, and C lasers, introduced by F. T. Arecchi, G. L. Lippi, G. P. Puccioni, and J. R. Tredicce, Opt. Commun. **51**, 3081 (1985), refer to single-mode laser dynamics ruled by one, two, or three equations, respectively. The three classes are associated with different time scales of the relaxation phenomena occurring in a laser.

[5]G. Broggi, A. Colombo, L. A. Lugiato, and P. Mandel, Phys. Rev. A **33**, 3635 (1986).

[6]C. Van den Broeck and P. Mandel, Phys. Lett. A **122**, 36 (1987).

[7]H. Zeghlache, P. Mandel, and C. Van den Broeck, Phys. Rev. A **40**, 286 (1989).

[8]R. Mannella, P. V. E. McClintock, and F. Moss, Phys. Lett. A **120**, 11 (1987).

[9]N. G. Stocks, R. Mannella, and P. V. E. McClintock, Phys. Rev. A **40**, 5361 (1989).

[10]P. Glorieux and D. Dangoisse, IEEE J. Quantum Electron. **QE-21**, 1486 (1985); E. Arimondo, D. Dangoisse, C. Gabbanini, E. Menchi, and F. Papoff, J. Opt. Soc. Am. B **4**, 892 (1987).

[11]W. Scharpf, M. Squicciarini, D. Bromley, C. Green, J. R. Tredicce, and L. M. Narducci, Opt. Commun. **63**, 344 (1987).

[12]A. Mecozzi, S. Piazzolla, A. D'Ottavi, and P. Spano, Phys. Rev. A **38**, 3136 (1988).

[13]M. C. Torrent and M. San Miguel, Phys. Rev. A **38**, 245 (1988).

[14]T. Erneux and P. Mandel, Phys. Rev. A **39**, 5179 (1989).

[15]F. T. Arecchi, W. Gadomski, R. Meucci, and J. A. Roversi, Opt. Commun. **70**, 155 (1989).

[16]F. O. Shimizu, K. Sasaki, and K. Ueda, Jpn. J. Appl. Phys. **22**, 1144 (1983).

[17]*Handbook of Mathematical Functions,* edited by M. Abramowitz and I. A. Stegun (Dover, New York, 1965).

[18]F. T. Arecchi, W. Gadomski, R. Meucci, and J. A. Roversi, Phys. Rev. A **39**, 4004 (1989).

[19]F. T. Arecchi, R. Meucci, and M. Ciofini (unpublished).