Periodic and Chaotic Alternation in Systems with Imperfect O(2) Symmetry

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Recent experiments on an optical oscillator with photorefractive gain have shown that the output field undergoes periodic or chaotic alternations among a small number of patterns. The above features are explained by a dynamical model, based only on symmetry arguments, and hence applicable to a variety of physical situations.

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Recent experiments on an extended optical system [1] have shown the sequential onset of a small number of different configurations, each one living for a rather long time and then being quickly replaced by another one. The system consists of a ring cavity with a photorefractive medium pumped by an argon laser. The cylindrical geometry of the cavity constrains the symmetry of the output field. The pumping process, however, breaks the O(2) cavity symmetry introducing a privileged plane, defined by the propagation vectors of pump and signal fields. The oscillator yields field patterns varying in time. By changing the size of the cavity pupil, two different dynamical regimes are observed. For large pupils the field displays a complex pattern which may be expanded in a large number of solutions of the free propagation problem (the so-called cavity modes). For small pupils, the field at any time is made of a single mode; however, a small number of modes (from two to about ten) can alternate in time. Thus, the alternation phenomenon consists of an ordered sequence of quasistationary modes. Depending on some control parameter, the persistence time of each mode is either regular [periodic alternation (PA)] or irregular [chaotic alternation (CA)]. Away from the narrow switching time intervals, the amount of mode mixing is negligible.

A phenomenon similar to CA, called chaotic itinerancy, was introduced by Ikeda, Matsumoto, and Otsuka [2], Otsuka [3], Kaneko [4], and Tsuda [5] in dealing with numerical solutions of different classes of model equations, namely, a one-dimensional laser [2], an array of coupled lasers [3], globally coupled iteration maps [4], and nonequilibrium neural networks [5]. In fact this latter phenomenon includes erratic jumps among the available quasistationary states, whereas CA keeps the sequence ordering. Such is the case in the experiment of Ref. [1], even though it was initially called chaotic itinerancy.

The purpose of this paper is to show that the onset of PA and CA is accounted for by pure symmetry arguments, without detailed knowledge of the underlying physics. Let us consider a simple dynamics involving three transverse modes, a central one with amplitude z_0 and two higher-order ones counterrotating along the azimuthal coordinate θ with respective amplitudes z_1 and z_2 and angular momenta ± 1. We can expand the cavity field as

$$E = f_1(r,l)(z_1 e^{i\theta} + z_2 e^{-i\theta})e^{i\omega_1 t} + f_0(r,l) z_0 e^{i\omega_0 t}, \quad (1)$$

where f_0 and f_1 are the space distributions of the modes. The optical frequencies ω_0 and ω_1 are in general different, and slow time dependence due to the dynamics is included in the amplitudes $z_i(t)$ $(i=0,1,2)$.

The zero intensity situation corresponds to $z_0 = z_1 = z_2 = 0$, the central mode to $z_1 = z_2 = 0$, and an azimuthal standing wave to $z_0 = 0$, $z_1 = z_2$. The time sequence of these three situations is one of the simplest cases experimentally observed [1], so we aim at building model equations having the above sets of z values as fixed points. Since, however, any quasistationary point persists for a finite time, each of the fixed points must have at least an unstable direction. With these general rules in mind we now discuss the symmetry requirements.

The observed symmetries impose the following constraints on the mode amplitudes [6]:

$$\Theta : (z_1, z_2, z_0) \to (e^{i\theta} z_1, e^{-i\theta} z_2, z_0),$$

$$K : (z_1, z_2, z_0) \to (z_2, z_1, z_0),$$

where Θ denotes the rotation operation, and K the reflection around the privileged plane. As these modes are born by Hopf bifurcations, there is an additional time symmetry

$$B : (z_1, z_2, z_0) \to (e^{i\beta_1} z_1, e^{i\beta_1} z_2, e^{i\beta_0} z_0).$$

The normal form for the nonlinear interaction among the three modes, assuming that the symmetry of the system is Z_2 (reflection) degenerated towards an O(2) one (reflection and rotation), is [7,8] (dots denote time derivatives)

$$\begin{aligned} \dot z_0 &= \lambda_0 z_0 + [a(|z_1|^2 + |z_2|^2) + b|z_0|^2] z_0, \\ \dot z_1 &= \lambda_1 z_1 + (c|z_1|^2 + d|z_2|^2 + e|z_0|^2) z_1 + \varepsilon z_2, \\ \dot z_2 &= \lambda_1 z_1 + (d|z_1|^2 + c|z_2|^2 + e|z_0|^2) z_2 + \varepsilon z_1, \end{aligned} \quad (2)$$

where $\lambda_0,\lambda_1,a,b,c,d,e$ are complex coefficients and $\varepsilon=\rho_\varepsilon e^{i\varphi_\varepsilon}$ is a symmetry-breaking parameter. Letting $z_i=\rho_i e^{i\varphi_i}$, and changing the variables as $\rho_1=A\cos(\alpha/2)$, $\rho_2=A\sin(\alpha/2)$, and $\delta=\varphi_2-\varphi_1$, Eqs. (2) will read

$$\begin{aligned}\dot{A}&=\{\lambda_1^r+[c^r-\tfrac{1}{2}(c^r-d^r)\sin^2(\alpha)]A^2\}A+[\rho_\varepsilon\sin(\alpha)\cos(\delta)\cos(\varphi_\varepsilon)+e^r\rho_0^2]A\,,\\ \dot{\alpha}&=-\tfrac{1}{2}(c^r-d^r)\sin(2\alpha)A^2+2\rho_\varepsilon[\cos(\varphi_\varepsilon)\cos(\delta)\cos(\alpha)+\sin(\varphi_\varepsilon)\sin(\delta)]\,,\\ \dot{\delta}&=-(c^i-d^i)A^2\cos(\alpha)+\rho_\varepsilon[\cotan(\alpha/2)\sin(\varphi_\varepsilon-\delta)-\tan(\alpha/2)\sin(\varphi_\varepsilon+\delta)]\,,\\ \dot{\rho}_0&=(\lambda_0^r+a^rA^2+b^r\rho_0^2)\rho_0\,,\end{aligned}\tag{3}$$

and

$$\dot{\varphi}_0=\lambda_0^i+a^iA^2+b^i\rho_0^2\,,\quad \dot{\varphi}_1=\lambda_1^i+A^2[c^i\cos^2(\alpha/2)+d^i\sin^2(\alpha/2)]+e^i\rho_0^2\,.\tag{4}$$

We will now show that the solutions of Eqs. (3) and (4) reproduce the experimental behavior for certain parameter values that we derive explicitly. Notice that Eqs. (3) constitute a closed four-dimensional system.

Both the laboratory experiment [1] and the numerical solution of the physical model [9] show that an initial condition close to a central mode is followed in time by a zero intensity state (see Fig. 1). Therefore, in the six-dimensional phase space of the solutions of Eqs. (3) and (4), the zero is stable in the ρ_0 direction. In our model, a first condition for the correspondence with the experiments is thus $\lambda_0^4<0$ and $b^r>0$.

If $\rho_0=0$ there are fixed points at $\alpha=\pi/2$, $\delta=0,\pi$, with

$$A^2=-2[\lambda_1^r\pm\rho_\varepsilon\cos(\varphi_\varepsilon)]/(c^r+d^r)\,.\tag{5}$$

These solutions correspond to standing waves, and therefore will be part of the "skeleton" of pure modes in our dynamical model.

The stability of these fixed points in the (α,δ) directions can be derived from the eigenvalues of the Jacobian of the first three Eqs. (3),

$$\begin{bmatrix}-2\lambda_1^r\mp2\rho_\varepsilon\cos(\varphi_\varepsilon) & 0 & 0\\ 0 & DG_{11} & DG_{12}\\ 0 & DG_{21} & DG_{22}\end{bmatrix},\tag{6}$$

where DG_{ij} stands for the 2×2 matrix

$$DG=\begin{bmatrix}\dfrac{2}{c^r+d^r}[(d^r-c^r)\lambda_1^r\mp4c^r\rho_\varepsilon\cos(\varphi_\varepsilon)] & \pm2\rho_\varepsilon\sin(\varphi_\varepsilon)\\ \dfrac{-2(c^i-d^i)}{c^r+d^r}[\lambda_1^r\pm\rho_\varepsilon\cos(\varphi_\varepsilon)]\mp2\rho_\varepsilon\sin(\varphi_\varepsilon) & \mp2\rho_\varepsilon\cos(\varphi_\varepsilon)\end{bmatrix}.\tag{7}$$

According to the experimentally observed dynamics, states close to zero evolve towards a standing wave. Therefore, the zero will be unstable in the $\rho_1=\rho_2=\rho$ direction, and that will occur if $-2\lambda_1^r\mp2\rho_\varepsilon\cos(\varphi_\varepsilon)>0$. As A^2 must be a positive quantity, according to Eq. (5) $c^r+d^r<0$. If the eigenvalues of DG have negative real parts, the fixed points will be stable in the α and δ directions also. The averaged intensity of these states can be computed as $\langle EE^*\rangle$, with E given by

$$E=e^{i(\varphi_1-\delta/2)}\rho[e^{i(\theta-\delta/2)}+e^{-i(\theta-\delta/2)}]f(r)e^{i\omega_1 t}\,.\tag{8}$$

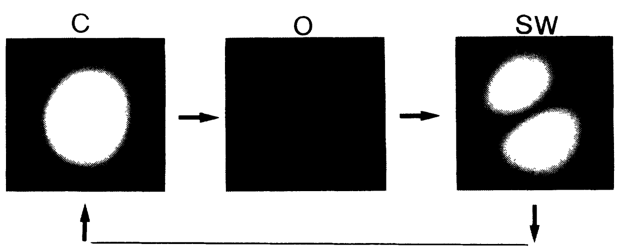

FIG. 1. Experimental sequence of three alternating patterns (from the data files of Ref. [1]). C is the central mode (only $z_0\neq0$), O is the zero mode ($z_0=z_1=z_2=0$), and SW (standing wave) is a balanced combination of clockwise (z_1) and anticlockwise (z_2) azimuthal traveling waves. The two spots of SW are aligned along the intersection with the privileged plane, corresponding to $\delta=0$. Each of the three patterns lasts around 100 sec and is replaced by the next one within 2 sec, along the sequence shown by the arrows.

Therefore, the pattern corresponding to this state will look like a set of two bright spots either parallel ($\delta=0$) or perpendicular ($\delta=\pi$) to the privileged axis as shown in Fig. 1.

Now let us state the conditions for switching to the central mode in such a way that the dynamics of our model closely resembles the experimental dynamics. Basically, we want the standing wave to be unstable in the ρ_1 and ρ_2 directions, so that initial states close to the standing wave and with small components of ρ_0 should evolve towards a central mode as shown in Fig. 2.

In the $\rho_1=\rho_2$, $\delta=0$ subspace, the dynamics is given by

$$\begin{aligned}&\dot{\rho}_0(\lambda_0^r+a^rA^2+b^r\rho_0^2)\rho_0\,,\\ &A=[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)+\tfrac{1}{2}(c^r+d^r)A^2+e^r\rho_0^2]A\,.\end{aligned}\tag{9}$$

In the (ρ_0,A) plane of the phase space there are three

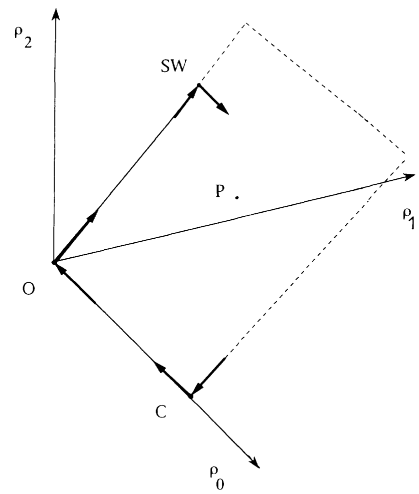

FIG. 2. Three-dimensional (ρ_0,ρ_1,ρ_2) projection of the phase space. The arrows indicate the local stability properties of the zero (O), central (C), and standing wave (SW) solutions, which lie on the (ρ_0,A) plane (sketched by dashed lines).

fixed points O, SW, and C, with respective coordinates (0,0), $(0,\{-2[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)]/(c^r+d^r)\}^{1/2})$, and $([-\lambda_0^r/b^r]^{1/2},0)$. If the following happens

$$\lambda_0^r-2\frac{a^r}{c^r+d^r}[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)]>0\,,$$
$$-\frac{e^r}{b^r}\lambda_0^r+[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)]<0\,, \tag{10}$$

the fixed point SW is unstable in the ρ_0 direction, and C is stable in the A direction. As $b^r>0$, C is unstable in the ρ_0 direction. Notice that $D\equiv a^re^r-b^r(c^r+d^r)/2$ must be negative in order for both conditions to be satisfied at the same time. If this happens, the (ρ_0,A) plane includes a fourth fixed point P with

$$\rho_0^2=\frac{1}{D}\left[\frac{c^r+d^r}{2}\lambda_0^r-a^r[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)]\right],$$
$$A^2=\frac{1}{D}\{-e^r\lambda_0^r+b^r[\lambda_1^r+\rho_\varepsilon\cos(\varphi_\varepsilon)]\}\,. \tag{11}$$

P undergoes a Hopf bifurcation if

$$\rho_0^2=-\left[\frac{c^r+d^r}{2b^r}\right]A^2\,. \tag{12}$$

As a periodic orbit emerging from P gets close to the other fixed points, there will be a critical slowing down that will give rise to the PA phenomenon among the O, SW, and C states.

The numerical solutions of the equations (Fig. 3) show a qualitative agreement with the above considerations. The parameter values chosen for the simulations are reported in the figure captions. Notice that in the case of Fig. 3(a), $b^r/e^r>2a^r/(c^r+d^r)$; thus the eigenvalues of P in the ($\alpha=\pi/2$, $\delta=0$) plane are $\pm i$. For any initial condition there is a periodic solution passing through it, suggesting the existence of integrals of motion. Notice that there is a heteroclinic solution connecting the three fixed points on the axes, and that the period of a periodic solution is larger the closer it is to the heteroclinic solution. Furthermore, the density of points along the trajectories

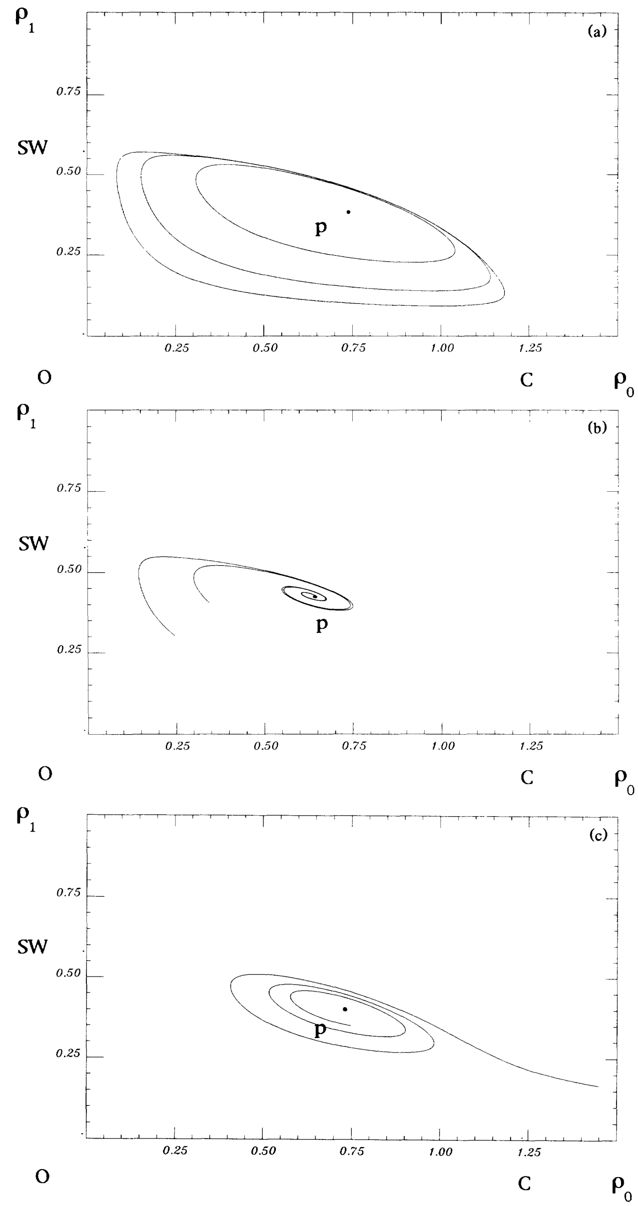

FIG. 3. (a) (ρ_0,ρ_1) projection of the solutions of Eqs. (3) for $a^r=3$, $b^r=1$, $c^r=-1$, $d^r=-2$, $e^r=-1$, $\lambda_1^r=0.5$, $\lambda_0^r=-1.5$, $\rho_\varepsilon=0.5$, $\varphi_\varepsilon=0$, and $c^i-d^i=1$. P is marginally stable and for any initial condition a periodic orbit is found. The closer a periodic orbit is to the three fixed points, the larger the period. The imaginary parts of the coefficients $\lambda_{0,1}$, b, a, and e contribute only to the dynamical evolution of φ_0 and φ_1. (b) Same for P stable [parameters such that $2b^r\rho_0^2+A^2(c^r+d^r)<0$]. (c) Same for P unstable [parameters such that $2b^r\rho_0^2+A^2(c^r+d^r)>0$].

shows a long persistence close to C, SW, and O and fast transitions in between, according to the experimental data of Fig. 4(b) in Ref. [1]. The above solutions should be compared with those corresponding to parameter values such that the fixed point P is stable [Fig. 3(b)] and unstable [Fig. 3(c)].

In the experiment of Ref. [1], a strong pulling action due to the narrow frequency width [10] of the photorefractive medium provides equal dressed frequencies for all transverse modes, even though the bare frequencies were different. Now, Eq. (1) refers to the modes dressed by their interaction with the medium; hence, a frequency degeneracy $\omega_1 = \omega_0$ should be included. A resonance between the states with angular momenta ± 1 and the central one implies an additional symmetry. Namely, the time symmetry becomes [6]

$$B:(z_1,z_2,z_0) \rightarrow (e^{i\beta}z_1, e^{i\beta}z_2, e^{i\beta}z_0) .$$

This allows additional terms to survive in the normal form equation, which will now read as follows:

$$\begin{aligned} \dot{z}_0 &= \lambda_0 z_0 + [a(|z_1|^2+|z_2|^2)+b|z_0|^2]z_0 + f z_1 z_2 z_0^* , \\ \dot{z}_1 &= \lambda_1 z_1 + (c|z_1|^2+d|z_2|^2+e|z_0|^2)z_1 \\ &\quad + \varepsilon z_2 + g z_0^2 z_2^* , \\ \dot{z}_2 &= \lambda_1 z_1 + (d|z_1|^2+c|z_2|^2+e|z_0|^2)z_2 + \varepsilon z_1 + g z_0^2 z_1^* , \end{aligned} \tag{13}$$

where f and g are complex coefficients. Dynamically, the additional terms due to resonance act as forcing terms with frequency $2\varphi_0 - (\varphi_1 + \varphi_2)$, thus inducing dramatic changes in the structure of the solutions. Notice that if the additional terms are "turned on" when the other parameters are tuned close to a heteroclinic solution, Melnikov-like arguments [11] suggest that some of the periodic solutions can disappear, some can bifurcate to solutions of different periodicity, and even chaotic behavior can be expected. Figure 4 reports the integration of Eqs. (13). Notice the existence of a chaotic solution getting close to the pure modes (CA). Both PA and CA phenomenon are structurally stable, insofar as they persist over wide ranges of parameter values.

In conclusion, using only symmetry arguments, we constructed a model for PA and CA in a dynamical system with a broken O(2) symmetry. We made reference to a particular system, but our study suggests that a similar behavior should be expected in other systems sharing the same symmetry properties. The essence of the mechanism is the following: The existence of a symmetry implies the existence of invariant planes in phase space. While in a nonsymmetric system heteroclinic connections are nongeneric, the symmetry constraints here considered make possible the existence of heteroclinic connections as well as periodic solutions close to them. When the system is in one of those periodic states there is a "periodic alternation." The effect of resonance is diverse and complicated; stable periodic solutions as well as chaotic ones (CA) can exist for a wide range of control parameters.

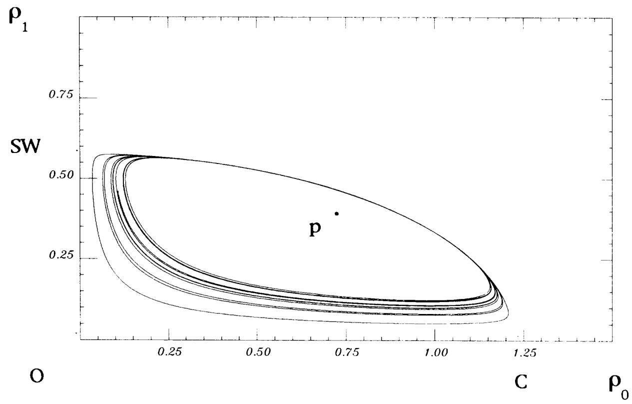

FIG. 4. (ρ_0,ρ_1) projection of a chaotic solution of Eqs. (13). Same parameter values as in Fig. 3(a) plus $f^r = -0.01$, $g^r = 0.02$, $f^i = 0$, and $g^i = 0$.

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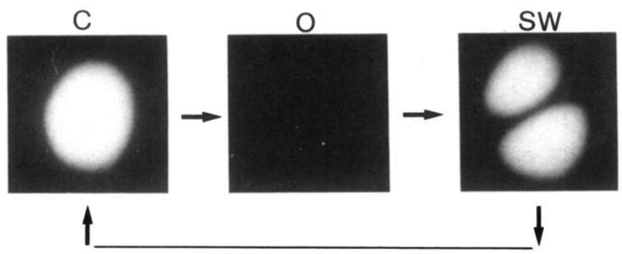

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