Optical Morphogenesis: Pattern Formation and Competition in Non-Linear Optics (*).

F. T. Areccchi
Dipartimento di Fisica dell’Università - Firenze
Istituto Nazionale di Ottica - Firenze

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Summary. — Pattern formation and competition occurs in a non-linear extended dynamical system if dissipation allows for attracting sets, independently of initial conditions. Most reported patterns are still dependent on boundary conditions. This dependency disappears in chemical Turing patterns emerging from reaction-diffusion dynamics. Also in diffraction-diffusion dynamics, as the one ruling an optical field coupled to a polarizable medium, similar phenomena can occur. A review of recent optical experiments is presented, including a discussion on suitable indicators which characterize the different regimes as well as the modelling of the various phenomena. Furthermore possible applications to quantum mechanics are sketched.

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1. – Introduction.

The onset of new patterns in space and time (morphogenesis) is the main stimulus for our curiosity. Indeed a uniform universe would be a very dull object. The variety of shapes and their evolution with time has been the most popular source of inspiration for poets and philosophers.

However, these phenomena have received scarce consideration in the program of modern physics, which, after Galileo and Newton, has been striving in search of simple explanations in terms of single, atomistic, components, leaving the complexity of patterns to qualitative descriptions not affordable in terms of the rigorous models.

Recent progress in non-linear dynamics has displayed some unifying features of morphogenetic phenomena. They occur in media which are away from thermal equilibrium, and require dissipative processes in order to cancel the memory of the initial conditions and let the system tend to an asymptotic state (attractor).

A further character is the existence of space gradients. Thus the modelling of these phenomena requires non-linear, dissipative partial differential equations (PDE).

A satisfactory understanding of morphogenesis in many areas represents a conceptual breakthrough. As examples, morphogenesis in fluid dynamics and in chemical reactions are crucial for all biological organization, not to say that the same distribution of celestial bodies is a morphogenetic problem within the gravitational field, and star formation as well as planetary evolution imply patterns in plasmas.

In the case of wave phenomena, as for the Maxwell field in electromagnetism or for the Schrödinger field in quantum mechanics, the conservative character of the wave equation seems to exclude morphogenesis, since there is no cancellation of the initial conditions and hence no attractors. However, coupling of a wave field with a dissipative medium breaks the above character and gives rise to morphogenesis.

Optical morphogenesis is a new interesting field of investigation for the following reasons: i) changing the size or the time scale of the experiment is much easier than in fluids or chemistry; ii) optical-pattern recognition has recently provided many tools for reliable detection and classification of space-time-dependent phenomena; iii) some of the optical phenomena may provide analogies with corresponding fluid phenomena (dry hydrodynamics); iv) the wave aspects suggest a «Gedanken-experiment» for the Schrödinger field; v) from a practical point of view, besides the progress in optical-pattern recognition (see ii)) which may be exploited in other areas, coding different messages via different patterns suggests the implementation of an optical parallel computer.

Since a recent, general review on pattern formation is available [1], I shall limit this paper to optical morphogenesis. Section 2 provides some classification criteria. Section 3 deals with passive optical systems, discussing how the pattern symmetry depends on boundary constraints. Section 4 deals with active optical systems, showing the large-volume limit where pattern self-organize, with sizes independent of boundaries. Section 5 provides an insight into the theoretical modelling of optical morphogenesis. Finally, in the conclusion I sketch morphogenesis in quantum systems coupled to a dissipative mechanism, through a «Gedanken-experiment» which is within reach of the nowadays available laboratory techniques. This experiment shows how the phase of the Schrödinger wave function undergoes a decoherence process as a microscopic system is coupled to a dissipative one.

2. – Morphogenesis and self-organization.

The first phenomenological treatments of morphogenesis were built for fluid dynamics, through the mathematical modelling of instabilities as those named after Faraday, Rayleigh and Bénard, Rayleigh and Taylor, Kelvin and Helmholtz etc.[2,3]. What is common to these phenomena is their dependence on gradients imposed at the boundaries, and hence the relation between their size and some geometric parameter of the confining volume. Take, e.g., the convective rolls in the Rayleigh-Bénard instability: their fundamental scale coincides with the height of the convective cell.

In this century, early tentatives of biological and chemical pattern modelling were based on the use of a diffusion equation, well known as responsible for pattern decay, plus a non-linear source term (Fisher equation[4]). But the real innovation was the Turing extension of non-linear diffusion to a vector field made of two components, one
being the activator and the other the inhibitor [5]. The concentration fields \( c_i(r, t) \) \((i = 1, 2)\) of the two species can destabilize out of the uniform solution toward time-independent patterns with a length scale of the order of \( (l_1 l_2)^{1/2} \) \((l_i = D_i \tau_i)^{1/2}\) is the diffusion length of species \( i \), having diffusion constant \( D_i \) and relaxation time \( \tau_i \) provided that \( \tau_i \equiv \tau_2 \) and \( D_1 \ll D_2 \). For different parameters, the fields can be uniform in space but time dependent (Hopf instability [6]). Finally Turing morphogenetic equations may also provide both space- and time-dependent phenomena (Turing-Hopf patterns) [7].

The main feature of Turing morphogenesis is that the patterns are independent of boundary parameters as well as of the geometry of the container, since the only parameters which matter are those appearing in the PDE (partial differential equations). In such a case, one can perform an infinite-volume limit, and consider patterns organized only by means of intrinsic parameters. This is an example of «self-organization» to be opposed to «hetero-organization», or pattern formation imposed from outside, as it usually occurs whenever the pattern sizes depend upon the gradients at the boundary, and thus are non-invariant to volume changes [8].

Only in this self-organized case it makes sense to draw analogies between morphogenesis and phase transitions in thermodynamic systems.

In the optical case, morphogenesis emerges from the interaction of an optical field, ruled by a diffractive equation, with a medium susceptibility ruled by a diffusion equation. Diffraction, being a coherent transport of a phase term over a given length, provides pattern sizes which depend on the cavity geometry [9]. The coupling with the diffusive medium has been so far considered in the thin-medium case (longitudinal size of the medium smaller than the diffusion length [10, 11]).

A further distinction to be introduced is that between passive and active optical morphogenesis. In the former case, an input field is distorted while propagating in a closed loop (cavity containing an optical medium). In the latter one, the intracavity medium is fed by an external pump and the corresponding excitation gives rise to the spontaneous emergence of a cavity field. Thus, passive morphogenesis consists in the distortion of an input field towards a new configuration, while active morphogenesis consists in the appearance of a field out of vacuum. The most studied passive system is a cavity with a Kerr medium within which a field undergoes a dephasing, and which has a source term proportional to the square of the field [10, 11]. This case has been dealt with in many experimental situations, including non-local interaction due to rotation and translation of the transverse optical pattern in the feedback loop [11].

For the active case, one class widely studied is that of laser phenomena, where the active medium is usually modelled as a collection of pumped two-level atoms. Another class, on which there is a wealth of experimental studies, is that of the photorefractive oscillators. In these systems the active medium is an array of dipoles made by free electrons and fixed donors. In the dark, the donor atoms are neutrals. As a pump and a signal field impinge on the medium, the large intensity corresponding to the interference peaks ionizes the donors yielding a space charge grating which scatters energy from the pump into the signal direction. Even in the absence of a macroscopic signal field (signal in the vacuum state), the creation operator of the signal field starts the process. As in the case of the lasers, even though the initial seed is quantum, a classical description is sufficient for all purposes. The space charge undergoes local relaxation (recombination processes), diffusion and drift in the presence of an applied d.c. field.
In the forthcoming sections, after a presentation of the phenomenology, we shall organize a theoretical framework along the following strategic lines, which provide a satisfactory agreement with the observation.

i) **Modelling the evolution by appropriate PDEs.**

The field equation is well established, indeed it is the wave equation in the diffractive limit of a carrier wave propagating in the z-direction at the optical frequency \( \omega \), so that second derivatives in z and time can be neglected [9,12], with full dependence on transverse coordinates \( x \) and \( y \), and with a source polarization depending on a medium susceptibility \( \chi(r, t) \). The material equation describes the evolution of \( \chi \); it includes local relaxation and diffusion plus a source term due to the non-linear coupling with the field. We limit the couplings to the two cases of a Kerr medium (sect. 3) and of a photorefractive medium (sect. 4).

We do not consider the more complex case of a laser medium, where the microscopic structure of each atom should be considered, and which brings to different model equations for different parameter regimes [13]. Furthermore, accounting for the parameter ranges accessible to the experiment, we refer to a precise type of non-linearity, and do not develop perturbative methods for extracting different types of equations appropriate for different regimes [14].

Finally, having the diffraction and diffusion Kernels mutually isolated, respectively in the field and in the medium equation, we do not need to consider an abstract prototype model as the complex Ginzburg-Landau equation, which combines both properties.

ii) **Linear stability analysis.**

Making a linear expansion around a stationary, uniform solution, the linear equations are Fourier analysed. This way, one establishes the threshold conditions, that is, the range of control parameters for which the uniform solution destabilizes, and the instability domains, that is, the bands of allowed \( q \) wave vectors (\( q \) being the transverse Fourier space corresponding to the \((x, y)\) space).

The linear treatment is particularly appropriate for a cavity of small size, where the boundary constraints select only a discrete number of possible modes within the allowed \( |q| \)-band. In such a case, a careful adjustment of a control parameter permits to observe single-mode behaviour.

iii) **Normal forms.**

Beyond the first bifurcation corresponding to the loss of stability of the fundamental solutions, high-order bifurcation leads to a coupling of the linearly independent modes. In the first instance, and whenever the number of allowed modes is reasonably small, one may consider a small set of ODE (Ordinary Differential Equations) describing the evolutions of the coupled mode amplitudes.

In a qualitative way the minimal coupling is that compatible with the symmetry constraints imposed by the dynamics [15].

Normal forms are a set of non-linear equations displaying all features of a dynamical system with a small number of degrees of freedom, such as limit cycles and chaotic oscillators. An apparently general feature of this regime is that regular or chaotic cycles correspond to heteroclinic connections among a small set of fixed points.
corresponding each to a single mode [16]. Along the cycle the velocity of the phase state point is strongly uneven, indeed, there is a long permanence close to each one of the fixed points, and a rapid passage to the neighbourhood of another one. Phenomenologically, this corresponds to a long persistence of a single mode, which is then replaced by another one, either with a regular or chaotic timing. We have called these two situations PA and CA (that is periodic and chaotic alternation), respectively [17].

iv) Statistics of phase singularities.

In pattern studies of fluids or chemical reactions, the number of competing modes is increasing by change of an intrinsic control parameter (e.g. increasing the excitation means a larger number of modes above threshold). An alternative way, peculiar of optics, is to increase the transverse size of the optical cavity and hence reduce the losses of a larger number of modes, at fixed excitation. As one increases the number of competing modes, the normal form treatment is no longer adequate, since many modes are present simultaneously. This situation is called STC (space-time chaos [17]). One then must resort to a statistical description in terms of space correlation functions. In fact, as many modes compete, their phase relations are changing, thus the transverse correlation length becomes smaller and smaller.

An alternative efficient description consists in detecting the phase singularity points, that is, those points around which the circulation of the phase gradient is non-zero. For a two-dimensional complex field \( E(r) \) this happens whenever \( \text{Re} E = \text{Im} E = 0 \) and hence \( \varphi = \arctg (\text{Im} E / \text{Re} E) \) is undefined.

The statistical distribution of these «dark points», and that of a topological indicator as the sign of the circulation (\( \pm 2\pi \)), called topological charge, permit a general characterization of these situations [18, 19].

In the case of a complex Ginzburg-Landau equation (CGL), these singularities coincide with topological defects, that is, with local structures at the edges of regions where different symmetries hold [20, 21].

In fact, the CGL has a two-dimensional space dependence, whereas phase singularities appear as standard features of a free three-dimensional optical field (in such a case their distributions is regular and it depends on the selected modes [22]), or in association with a diffusively thick material, as discussed in sect. 4. That is why in general one cannot identify phase singularities with defects.

v) Noise and power spectra.

As the number of coexisting modes increases and the correlation length decreases, there is a sharp change in the scaling properties of phase singularities. This change corresponds to a transition from a regime dominated by the boundaries (i.e. hetero-organized) to a regime dominated by the bulk properties of the medium (i.e. self-organized) [23].

In such a case, the only global characterization which makes sense is the measurement of the spatial power spectrum. A heuristic way for evaluating it theoretically is the following. In the presence of a very large number of modes, the dominant ones can be still approximately treated by linear equations, but with a noise source accounting for the collective non-linear perturbation due to the other ones (self-induced mode partition noise, as called by Ikeda et al. in a one-dimensional model of a laser [24]).
In such a case the experimental correlation range is very short, thus one considers a white-noise source, which provides a power spectrum qualitatively similar to the experimental one.

vi) Non-local feedback.

Optical morphogenesis requires a cavity confinement, that is, a feedback in order to keep a tight interaction between optical field and medium susceptibility. In the feedback loop one can insert interactions which are non-local either in time or space.

Low-dimensional chaos due to time-delayed feedback has been suggested [25] and observed [26] long ago. Space-shifted feedback (i.e. reintroduction of a rotated pattern in the feedback channel) has been introduced by the Moscow group [11].

In sect. 3, we shall see in detail the role of space non-locality. Recently, my group has actively investigated the role of time delays on morphogenesis, but in this review there is no space for a detailed report on these cases, and I provide some information in a footnote [27].

In conclusion, there is a large gap between the detailed modelling of regular patterns, reducible to one, or a few, eigenmodes of the free propagation problem (ii), (iii)) and the qualitative description of the high-dimensional case (iv) to vi) where a general framework is still lacking. Possibly, suitable indicators easy to calculate and with a high degree of predictability will arise from discretized evolution models, in terms of Coupled Map Lattices or Cellular Automaton, rather than PDEs.

3. - Passive optics-feedback on a Kerr medium.

In this section we discuss the interaction of an optical field in feedback configuration with a passive medium where the susceptibility depends on the field intensity (Kerr medium). A series of pioneering experiments has been carried in resonant alkali vapours [28-30]. Here, however, we focus the attention on a thin slice of liquid-crystal (LC) molecules. In such a case hexagon formation has been predicted [10] and observed experimentally [31,32].

A clever amplification system, introduced by the Moscow group [11,33] allows morphogenesis with threshold intensities of a few mW/cm². Exploiting such an idea, we report some recent experimental advances, namely:

i) roll-hexagon competition, with plane-wave input and 180° azymuthal rotation of the feedback wavefronts [34],

ii) evidence of lower symmetry classes $D_l$ ($l = 2$ to 7) for a transversely confined input [36,37],

iii) two-dimensional crystals and quasi-crystals by azymuthal rotation of the feedback wavefront [38].

The system consists of a liquid-crystal light valve (LCLV) in a ring optical cavity. The LCLV is a mirror sandwiched between a nematic-liquid-crystal (LC) layer and a photoconductive layer. An AC sinusoidal voltage is applied across the photoconductive and the LC layers. In the usual operating conditions, the r.m.s. voltage amplitude is about 20 V and its frequency is 20 kHz. The liquid-crystal director (axis of the aligned LC molecules) is vertical as well as the input beam polarization. In the absence of light, most of the voltage drops across the photoconductive layer. The
rest, applied to LC, tends to orient the molecules longitudinally, thus changing the refractive index. In the presence of illumination the photoconductive layer has a smaller voltage drop, the voltage applied to LC increases and further reorientation of the molecules induces a phase shift of the reflected beam proportional to the intensity $I$ of the beam incident on the back of the LCLV. In this way the system acts as a Kerr-like medium. The non-linearity is defocusing because the LC molecules tend to align longitudinally, thus decreasing the refractive index along the propagation direction.

The advantage of using the LCLV instead of a simple layer of LC is that the non-linear phase shift requires a low input intensity (around 5 mW/cm$^2$) and can be controlled by changing the applied voltage.

3'1. Roll-hexagon competition. – The experimental set-up is shown in fig. 1a). The He-Ne laser input beam is expanded and collimated by the lens system $L_0$ and $L_1$. The beam retroreflected by LCLV is sent through the feedback loop via the beamsplitter $BS_1$ and the mirror $M$. $L_1$ and $L_2$ have the same focal length ($f = 25$ cm) and their separation is equal to 2$f$. Lens $L_2$ provides a confocal configuration whereby an image of the LCLV plane is formed, at a distance $f$, on the image plane $z_i$.

A fibre bundle is connected to the LCLV. The distance $L$ between the end of the fibre bundle and the plane $z_i$ is a free propagation range where diffraction is effective. Diffraction converts phase into amplitude modulation.

When the fibre bundle is twisted to produce a 180° image rotation, the resulting pattern has an inversion symmetry around a centre. Hence, a pair of points sym-

![Fig. 1. – a) Passive optical morphogenesis–experimental set-up; $L_i =$ lens; $M =$ mirror, $BS =$ beamsplitter; $PH =$ pinhole. b) Schematic representation of the effect of 180° feedback rotation; left: two round trips are needed in order to map each point onto itself; right: this is equivalent to considering two coupled virtual systems.](image-url)
Fig. 2. – Experimentally observed patterns: a) rolls, b) hexagons, c) hexagons + rolls and d), e), f) corresponding far-field patterns. On the top we report for each of the three situations a diagram of the feedback intensity $I$ vs. transverse position $x$ (normalized between 0 and 1). We calibrate the average intensity (dashed line) and the offset (solid line), with reference to the control parameters used in the stability diagrams of ref. [35].

metrical with respect to the centre are transformed one into the other in a single round trip, and all points of the pattern are mapped onto themselves after two round trips. A similar behaviour occurs when there is a reflection symmetry. In these cases the system is equivalent to two virtual systems coupled by their mutual feedback (see fig. 1b)).

When the feedback intensity is the same on the two virtual systems, we have stable rolls (fig. 2a)) at variance with ref. [10]. In order to unbalance the coupling between the two virtual systems, we insert a neutral density filter (which attenuates the feedback intensity) only on one-half of the fibre bundle free end. In this case we have stable formation of hexagons of opposite polarity (fig. 2b)). We call positive (negative) hexagons $H^+$ ($H^-$) the ones on the right (left) half of the image. If the intensity change is not sharp but smooth as on the trailing edge of a Gaussian beam profile, then hexagon and rolls coexist in different regions of the wave front (fig. 2c)).

In the case of $0^\circ$ rotation of the optical feedback the only steady state that we
observe is a pattern of positive hexagons. Indeed, this situation corresponds exactly to the model of ref. [10].

32. \( D_l \) symmetries imposed by a confined input. — For the case of a Kerr medium with a feedback mirror, the solutions carrying the full symmetry of the system bifurcate to solutions with a lower symmetry (corresponding to a subgroup of the full symmetry) when boundary limitations are imposed [35]. Therefore, while with plane-wave illumination, patterns are invariant under translations, reflections and rotations in the transverse plane \((E_2 \text{ symmetry group})\), and hence rolls and hexagon patterns can be stable, illumination by a confined Gaussian input beam forces the system to be invariant only under rotations and reflections in the transverse plane \((O_2 \text{ symmetry group})\). This provides stable solutions with an overall dihedral symmetry \( D_l \) \((l \text{ being an integer})\) corresponding to a subgroup of \(O_2\). In this case spatial patterns are characterized by \(l\) axes of symmetry, according to the \(D_l\) group selected by the control parameters. Though preliminary results in this sense have already been given in ref. [36], a complete set of experimental observations was given in ref. [37] based only on phase shift effects, whereas in ref. [36] polarization effects make the system equivalent to a non-linear interferometer.

The experimental set-up is as described above. In order to check the role of the boundary conditions we insert an aperture on the optical-feedback path. Precisely, a variable aperture \(A\) is placed in the image plane \(z_i\) of lens \(L_2\), providing a confinement of the transverse active region. This way, although the field should be fragmented into hexagons due to the Kerr cubic non-linearity, it is instead forced to fit the aperture area, thus adjusting to the subgroup of the original symmetry allowed by the boundary.

The total diameter of the LCLV is 2 cm, whereas the diameter \(d\) of the diaphragm \(A\) in the feedback loop usually is varied from 1 to 2.2 mm. All the other parameters of the experiment are kept fixed. Starting from \(d = 1\) mm and increasing the aperture in the above range, we observe the subsequent appearance of patterns with symmetry \(D_l\) with \(l\) equal successively to 3, 2, 4, 5, 6, 7. Near-field images of these patterns are shown in fig. 3.

The patterns corresponding to \(D_l\) \((l = 2, 3, 4, 5, 6, 7)\) have \(l\) symmetry axes. In most cases this corresponds to \(l\) bright spots arranged at the edges of a polygon of \(l\) sides. The number of spots increases with the aperture, while the separation between spots remains fixed at a value very close to the instability length \(\Lambda = \sqrt{2L}\). Only very small adjustments around \(\Lambda\) are allowed, due to the fact that we are working very close to the instability threshold. The observed patterns are stationary, but it is worth noticing that they are very sensitive to misalignments so that only a careful adjustment of the optical elements can provide such stable structures.

Besides stationary configurations, we also observe secondary bifurcations which lead to dynamical behaviour of the patterns. Indeed, what we observe is that, changing by small amounts some control parameters (e.g. alignment, aperture diameter, frequency and/or amplitude of the voltage applied to the LCLV), in some cases the pattern starts to rotate, in other cases there is an alternation between two patterns corresponding to two successive \(D_l\) groups, and in some other cases the pattern oscillates coherently in time.

Furthermore, by setting the aperture at the diameter \(d = 1.8\) mm, intermediate between the stable operation of the \(l = 5\) and \(l = 6\) patterns, and carefully adjusting the r.m.s. amplitude of the sinusoidal voltage applied to the LCLV, we reach a
regime of periodic alternation between the above patterns. Periodic alternation (PA) was first observed in active optics (see sect. 4) and a dynamical description in terms of a heteroclinic cycle connecting the representative points in phase space was given in ref. [16]. A model for alternation will be sketched in sect. 5.

3'3. Optical crystals and quasi-crystals. – As shown in sect. 5, at the linear stage a circle of critical transverse wave vectors $|q|$ become simultaneously unstable. Close to threshold, the pattern is determined by non-linear interactions among wave vectors on this circle which has a radius $q_{\perp} = \sqrt{\pi k_0 / L}$ for a focusing medium and $q_{\parallel} = \sqrt{3\pi k_0 / L}$ for a defocusing medium, $k_0 = 2\pi / \lambda$ being the optical wave number, and $L$ being the free propagation length.

The free end of the fiber bundle is mounted on a rotation stage which allows for a continuous angular positioning over a full $360^\circ$ range with a high resolution (readout to $0.2^\circ$). This way, the feedback image arriving at the photoconductive layer of the LCLV can be rotated by any angle $\Delta$. As discussed in subsect. 3'1 for $\Delta = 0$ only hexagons are stable. On the other hand, for $\Delta = \pi$ it was shown that only rolls are stable and that a competition between hexagons and rolls can be achieved inserting an attenuation filter in front of one-half of the LCLV (previous subsect. 3'1).

As the rotation angle is exactly commensurate to $2\pi$, that is, when $\Delta = 2\pi / N$ with $N = 2, 3, 4, ...$, a pattern with an $N$-fold symmetry develops. The symmetry is induced by the rotation angle in the feedback and it is not due to boundary effects that arise when the system is strongly limited in its transverse extension (subsect. 3'2).

Originally, these rotation-induced patterns were discovered by the Moscow group and called «Akhseals» in memory of Serghiei Akhmanov (see ref.[11c]. Refer-
ence [38] is a systematic experimental investigation of these symmetries, with a quantitative modelling.

In fig. 4 we report near-field patterns, obtained for various rotation angles $\Delta = 2\pi/N$, where $N$ is changed from 2 to 9. Since the critical wavelength is of the order of 1 mm and the LCLV has a diameter of 3 cm, a central region with negligible boundary influences has been observed up to $N$ of the order of 20. Looking at the edge, it can be seen that away from the center the system tends to stabilize rolls for $N$-even and hexagons $N$-odd. In fact, the feedback rotation constraint is more efficient close to the center, whereas at the edge the patterns recover the two basic symmetries corresponding to $\Delta = 0$ ($N$ odd) and $\Delta = \pi$ ($N$ even).
Fig. 5. – Optical crystals and quasi-crystals: far-field patterns; all frames correspond to the same magnification, thus the rings with $N$ even and those with $N$ odd are respectively proportional to $q_1$ and $q_2$ (first and second instability peaks, as discussed in sect. 5).

The far-field patterns (fig. 5), collected on the focus of a lens, are the power spectra of the corresponding near-field images, and provide directly the number of modes involved in the pattern formation and the length of the critical wave vector (radius of the ring where the peaks are located). The fact that for any pattern each component contributes as a pair of opposite directions gives rise to $2N$ peaks for $N$ odd, but only to $N$ peaks for $N$ even.

A linear-stability analysis explains the presence of two different critical wave vectors when a rotation is introduced in the feedback path (see sect. 5).
4. – Active optics: photorefractive oscillators.

4'1. PA, CA, STC (periodic and chaotic alternation, space-time chaos) [17]. – We seed a ring cavity with a photorefractive gain medium pumped by an argon laser and study the temporal and spatial features of the generated field. By varying the size of a cavity aperture, it is possible to control the number of transverse modes which can oscillate. The experimental set-up consists of a ring cavity with gain. The gain medium is a photorefractive $5 \times 5 \times 10$ mm$^3$ BSO (bismuth silicon oxide) crystal to which a d.c. electric field is applied. The crystal is pumped by a c.w. argon laser with an intensity around 1 mW/cm$^2$.

The Fresnel number of the cavity is controlled by a variable aperture. The Fresnel number gives the maximum order of the transverse modes that can oscillate. $F$ can be varied in the range from 0 to approximately 100. Correspondingly, the total number of transverse modes allowed by diffraction scales as $F^2$.

Transverse intensity patterns corresponding to increasing Fresnel numbers $F$ are shown in fig. 6 together with fluctuations around the local average. The low $F$ limit ($F \leq 4$) corresponds to a time alternation among pure cavity modes (fig. 6a)), yielding a spatial correlation length $\xi$ of the order of the transverse size $D$ of the beam. For high $F$ ($F \geq 10$), on the contrary, the signal appears as a speckle-like pattern irregularly evolving in space and time (fig. 6c)), with a short correlation length ($\xi/D < 0.1$). The transition between these two limits is characterized by a continuous variation of the ratio $\xi/D$. A generic intermediate situation is shown in fig. 6b). An example of alternating pure mode configurations is given in fig. 7.

In order to study the time behaviour of the system we perform a local measurement of intensity vs. time by placing an optical fiber in an arbitrary point on the wavefront, so that each mode is encoded by a local intensity level. Identifying each mode with its azimuthal quantum number (defined as half the number of nodes along the circumference), we have modes 7, 6, 5, 4, 3, 2, 1, 0 alternating on a time scale of seconds, that is, of the order of the dielectric relaxation time of BSO (fig. 8).

The alternation is peculiar to low Fresnel numbers. It is observed down to the minimum $F$ for which the cavity can oscillate. While for large apertures the field pattern may be expanded in a large number of cavity modes, for small apertures the field at any time is made of a single mode, however a small number of modes (from two to about ten) alternate in time. Thus, the alternation phenomenon consists of an ordered sequence of quasi-stationary modes. Depending on some control parameter, the persistence time of each mode is either regular (periodic alternation: PA) or irregular (chaotic alternation: CA). Away from the narrow switching time intervals, the amount of mode mixing is negligible.

A phenomenon similar to CA, called chaotic itinerancy, was introduced by several authors in dealing with numerical solutions of different classes of model equations, namely, a one-dimensional laser [24], an array of coupled lasers [39], and globally coupled iteration maps [40]. In fact itinerancy includes erratic jumps among the available quasi-stationary states, whereas CA keeps the sequence ordering.

Increasing the value of the control parameter, we enter a new regime, called spatio-temporal chaos (STC) where a large number of modes coexist. This regime is characterized by a short correlation range defined as follows. Suppose we have a generic field $u(r,t)$ ruled by a partial differential equation including non-linear and
Fig. 6. – Photorefractive oscillator in a ring cavity (ref. [17]): intensity distribution of the wavefront (left) and spatial autocorrelation function (right) for increasing Fresnel number. 

a) $F = 1$, one single mode at a time is present, ratio between coherence length $\xi$ and frame size $D$ is $\xi/D = 1$; 
b) $F = 7$, $\xi/D = 0.2$; 
c) $F = 10$, $\xi/D = 0.06$. 
gradient terms. Let us consider the deviations away from the local time average

\( \delta u(r, t) = u(r, t) - \langle u(r, t) \rangle \),

where \( \langle \ldots \rangle \) denotes time average. Under very broad assumptions, we can take the leading part of the correlation function as an exponential, that is,

\( C(r, r') = \langle \delta u(r, t) \delta u(r', t) \rangle \equiv \exp[-|r - r'|/\xi]. \)

Whenever the correlation length \( \xi \) is larger than the wavefront size \( a(\xi > a) \) we have low-dimensional chaos, that is, even though the system can be chaotic in time, it is coherent in space (single mode, in a suitable mode expansion). The corresponding chaotic attractor is low-dimensional. In the limit \( \xi \ll a \), a local chaotic signal is not confined in a low-dimensional space and we speak of STC.

4.2. **Phase singularities.** – A phase singularity is a point around which the circulation of the phase gradient is a multiple of \( \pm 2\pi \). We call topological charge the multiplicity number. In the case of the wave equation only \( \pm 1 \) charges are stable. The nature of these singularities is determined by the fact that \( E \) is a smooth single-valued function of its variables. Single-valuedness implies that during a circuit \( C \) in space-time \( \phi \) may change by \( 2m\pi \), where \( m \) is an integer. Suppose \( m \) is not zero and let \( C \) be shrunk to a very small loop in such a way that \( m \) does not change. Then \( C \) encloses a singularity because \( \phi \) is varying infinitely fast. The smoothness of \( E \) now
Fig. 8. – Periodic alternation in a photorefractive oscillator. Time records of local intensity at $F = 5$ (from ref. [17]).

implies that this can happen only where $E = 0$, i.e. where $\varphi$ is indeterminate. Since the vanishing of $E$ requires two conditions (Re $E =$ Im $E = 0$) these phase singularities are lines in space or points in the plane (fig. 9a)). Sometimes we call the phase singularities as «wavefront dislocations» or, by analogy with fluid dynamics, vortices.

Let us consider a random field $E(x, y)$. If it is formed by the interference of a large number of independent components, then Re $E$ and Im $E$ are two independent random functions with Gaussian statistics.

The zeroes of the function Re $E(x, y)$ determine a number of curves in the $(x, y)$ plane, see fig. 9b). There is another set of curves corresponding to Im $E(x, y) = 0$, and now the intersections of curves of one family with those of the other give discrete points where $|E(x, y)| = 0$. If we consider the problem of the propagation of such zeroes along the direction $z$ in accordance with the wave equation, the discrete points in the $(x, y)$-plane are converted into lines. It is clear that in general this lines do not intersect in three-dimensional space. Moreover, a given line cannot appear singly at some plane $z = \text{const}$, nor can it disappear singly. Zeroes in the pattern must appear or be annihilated in pairs.

Up to now we spoke of a random speckle pattern. It is clear, however, that the topological dimensional arguments do not depend on the nature of the interfering fields. The only difference is that for regular fields, zeroes may not occur at all.

The difference of $n_+$ and $n_-$, the numbers of zeroes with positive and negative charges, is conserved in the process of propagation. On the average in a cross-section of a speckle field $n_+ = n_-$ since the beam is statistically homogeneous. The dislocation density coincides with the number of speckles per unit area.
verified in a series of experiments [41, 42], where the speckle field was obtained by transmitting a laser beam through a distorting phase plate. The structure of the density $N$ scales like $a^2$ ($a$ is the diaphragm diameter on the phase plate) and thus linearly in $F$; thus the total number of dislocations $Na^2$ scales like $F^2$.

In non-linear optics we have recently shown experimental evidence of phase singularities [18]. Their positions and the scaling of their separations, number and charges with the Fresnel number allows a classification of patterns.

At variance with the material waves which are easily visualized in terms of matter displacements, in the case of an optical field a phase measurement requires heterodyning against an external reference. Phase information is extracted by beating the signal with a reference beam onto a CCD video camera.

By a suitable algorithm, we reconstruct the instantaneous surfaces of phase as
shown in fig. 10 where the phase surface of a doughnut mode is a helix of pitch 2 around the core (vortex).

When more than one vortex is present, in order to resolve and count each vortex, we tilt the reference beam so that the video signal is now given by

\[ I(x, y) = A^2 + B^2 + 2AB \cos (Kx + \phi(x, y)), \]

where \( A \) and \( B \) are the amplitude of reference and signal field, \( K \) is the fringe frequency due to tilting, \( x \) is the coordinate orthogonal to the fringes and \( \phi \) is the local phase. This way, a phase singularity appears as a dislocation, and the topological charge is visually evaluated.

We digitize the fringe system and count those defects separated by at least one fringe, in the region where fringes can be resolved. Figure 11a) shows a configuration with an overall unbalance in the topological charge. A heuristic explanation of fig. 11a) is that, for small \( F \), the dynamics is strongly boundary dependent. Consequently we expect that an increase of \( F \) should eventually yield the thermodynamic limit of

![Fig. 11. Two examples of experimental configurations obtained by digitizing the fringe maxima (from ref. [18]). a) \( F = 3 \), six defects of equal topological charge against one of opposite charge. b) \( F \sim 10 \): about 70 defects of opposite charges, with residual small charge unbalance.](image)
paired charges. This is indeed the case as shown in fig. 11b), which refers to a high $F$ and where the charge unbalance $U = |n_+ - n_-|$ has become very small compared to the total number $N = n_+ + n_-$ of charges.

By averaging over a large number of frames for each $F$, we can find $\langle N \rangle$, the mean number of singularities per frame, and $\langle D \rangle$, the mean nearest-neighbour distance.

![Graph](image)

Fig. 12. – Scaling of vortex statistics with the cavity size (ref. [23]). a) Mean nearest-neighbor separation $\langle D \rangle$ (scale in μm) between phase singularities and b) average total number $\langle N \rangle$ of phase singularities vs. the Fresnel number $F$ of the cavity. Dashed lines are best fits with the boundary-dependent scaling laws $\langle D \rangle = F^{-1/2}$ and $\langle N \rangle = F^2$. Solid lines are best fits with the bulk-dominated scaling laws $\langle D \rangle \approx F^0$ and $\langle N \rangle \approx F^1$. 
These are plotted as a function of $F$ in fig. 12. Notice that, for low $F$, $\langle N \rangle$ and $\langle D \rangle$ have a power law dependence on $F$ with exponents close to $+2$ and $-0.5$, respectively.

Figure 13 gives $U_N$, the excess $U$ normalized to the total singularity number $N$. For small apertures the dynamics is strongly boundary dependent and the excess is large. For increasing $F$, $U_N$ decreases as a power law with exponent close to $-1.5$.

The spatial disordering of singularity positions is associated with the passage to STC. Since a phase singularity must be associated with a zero crossing of real and imaginary parts of the field, it follows that all intensity zeroes are singularities. But the diffractive treatment of optical cavities shows that the number $N$ of intensity zeroes for the highest allowed mode scales as the square of $F (N \sim F^2)$. On the other hand, if $a$ is the pupil aperture of the optical system, and $D$ is the average inter-defect separation, we expect $N \approx a^2 / D^2$, and, since $a^2 \approx F$, then $D \approx F^{-0.5}$. Such scaling laws are verified in fig. 14. We can justify also the $F$-dependence of $U_N$. Assume that unpaired defects are mainly created at the boundary, while in the bulk pairs with compensated charge are created and destroyed. Then the total number $N_c$ of boundary defects in the perimetal region of area $a \cdot D$ scales as $N_c \approx aD / D^2 \approx a / \langle D_i \rangle \approx F^{1}$, and the corresponding unbalance is $U \approx \sqrt{N_c} \approx F^{0.5}$. Hence the normalized unbalance scales as $U_n \approx F^{0.5-2} = F^{-1.5}$, in accord with the experiment.

$F$ scalings of the dislocation numbers and of their mean separation roughly equivalent to those reported in fig. 12 are also found for linear dislocations [42]. The most crucial test of the non-linear nature of the photorefractive dislocations is given by their time dependence. We select a small box of side $\xi$ (a correlation domain) where generally there is zero or one defect present, and measure the occurrence time of events, where an event is the entrance of a defect into the

---

**Fig. 13.** – Average charge imbalance $U \equiv \langle |n_+ - n_-| / (n_+ + n_-) \rangle$ vs. $F$. $n_+$ ($n_-$) is the total number of positive (negative) phase singularities. Dashed line: $F^{-1.5}$ fit up to $F = 11$; solid line: $F^{-0.75}$ fit from $F = 11$ on. Inset: expanded view of the high-$F$ region (from ref. [23]).
Fig. 14. – Field power spectrum, measured integrating the signal intensity over concentric shells of radius $q$ (mm$^{-1}$) in the Fourier space. Dots: experimental points, $F = 70$. Solid line: best fit (from ref. [23]).

box. This way, we build a sequence of time intervals, each defined by two successive events.

In ref. [18] the corresponding mean separation, $\langle T \rangle$, is plotted against $F$ and the pump intensity $P$. Since for any setting of $F$ and $P$ the time $\langle T \rangle$ is of the same order of the long time scale that characterises the mode competition, we infer that mode jumping is mediated by the vortex dynamics, as expected from the theory [20,21].

4'3. Transition from boundary- to bulk-controlled regimes. – As we increase the Fresnel number, we see the transition from patterns dominated by the geometric parameters, to dissipative patterns whose scale length is imposed by the bulk properties of the medium.

A fundamental geometric parameter is the spot size of the central mode which is constrained by the quasi-confocal configuration. The spot size of the central mode is given by

$$w_0 = \frac{\lambda L}{\pi}.$$  \hspace{1cm} (3)

Provided the mirror size $a$ is larger than $w_0$, that is, that the Fresnel number $F = \frac{a^2}{\lambda L}$ is larger than 1, the cavity can house higher-order modes, made of regular arrangements of bright spots (peaks of Gauss Laguerre functions in cylindrical geometry) separated by

$$D \equiv \frac{w_0}{\sqrt{F}}.$$  \hspace{1cm} (4)

Since the overall spot size of a transverse mode of order $n$ scales as $\sqrt{nw_0}$, it is clear
that $n = F$ is the largest-order mode compatible with the boundary conditions (filling all the aperture area). Notice that patterns built by superposition of Gauss-Laguerre functions have in general an average separation $\langle D \rangle$ of zero intensity points approximately equal to the average separation $D$ of bright peaks [43].

Extending the range of the explored $F$ values, the plot of $\langle D \rangle$ vs. $F$ (fig. 12) shows that eq. (4) is verified up to a critical value $F_c$, above which $D$ is almost independent of $F$. In a similar way, the total number $N$ of phase singularities scales as $F^2$ or $F$, respectively, below and above $F_c$. This transition is evident in fig. 12b) and its root is the following. $N$ is the ratio of the total wavefront area $a_2$, which scales as $F$, to the area $\langle D \rangle^2$ containing a single defect, which scales as $F^{-1}$ or $F^0$, respectively below and above $F_c$.

We can understand the transition as follows. Assume that the photorefractive crystal is a collection of uncorrelated optical domains, each one with a transverse size limited by a correlation length $l_c$ intrinsic to crystal excitations. Then the medium gain has an upper cut-off at a transverse wave number $1/l_c$ and spatial details are amplified only up to that frequency, that is, provided they are bigger than $l_c$. Thus, for a critical $F_c$ such that $D = w_0/\sqrt{F_c} = l_c$ we expect a transition from a boundary to a bulk-dominated regime where the separation of the phase singularities is independent of $F$. This is indeed the case as shown in fig. 12 which yields a value $F_c \approx 11$ corresponding to $l_c \approx 170 \mu m$, since $w_0 \approx 600 \mu m$ for $L = 200 cm$.

The reduction of the boundary influence is also signalled by the reduction of the topological charge imbalance. Indeed, since a regular field should have a balance between topological charges of different sign, an imbalance means that two phase singularities of opposite sign have been created close to the boundary and only one has remained within the boundary. Therefore, there is a boundary layer of area $a \langle D \rangle$ containing $N_1 = a/\langle D \rangle$ singularities. The absolute value of the imbalance is, for statistical reasons, of the order of $\sqrt{N_1}$, and thus the normalized imbalance $U = \sqrt{N_1}/N$. Accounting for the scaling of $\langle D \rangle$ with $F$, it follows that $U$ scales as $F^{-1.5}$ and $F^{-0.75}$, respectively below and above $F_c$.

A further independent check of bulk-dominated patterns is given by the power spectrum of the transverse optical field for $F > F_c$. This measurement is done by integrating the signal intensity over concentric shells of radius $q$ in the Fourier space provided by the far-field propagation of the cavity field. The results are reported in fig. 14. The best fit of the data yields an exponential high-frequency cut-off $\exp(-q/q_0)$ with $q_0 = 53 mm^{-1}$. This corresponds to a correlation length for the field, in good agreement with the values of $l_c$ reported above.

A broad-band spectrum with an exponential cut-off is a clear signature of spatio-temporal chaos, where a chaotic dynamics of spatial patterns is involved, with a dominant length scale.

5. – Modelling optical morphogenesis.

51. General features. – Optical morphogenesis came up later with respect to other types of pattern formation, because all coherent optics was based on the idea, basic for lasers, of a drastic mode selection, down to a single mode with an amplitude uniform in space.

Since optical laboratory phenomena are confined in a cavity mainly extended in a $z$-direction (as e.g. the Fabry-Perot cavity), we expand the field $e(r, t)$, which obeys
the wave equation

\[ \Box^2 e = -\mu \partial_t^2 p \]

(we simplify derivatives as e.g. \( \partial_t^2 \) rather than \( \partial^2 / \partial t^2 \), and \( p(r, t) \) is the induced polarization), as

\[ e(r, t) = E(x, y, z, t) \exp[-i(\omega t - k z)]. \]

If the longitudinal variations are mainly accounted for by the plane wave, then we can take the envelope \( E \) as slowly varying in \( t \) and \( z \) with respect to the variation rates \( \omega \) and \( k \) in the plane-wave exponential. Furthermore we call \( P \) the projection of \( p \) on the plane wave. By neglecting second-order envelope derivatives it is easy to approximate the operator on \( E \) as

\[ \Box^2 \rightarrow 2ik \left( \partial_z + \frac{1}{c} \partial_t \right) + \partial_x^2 + \partial_y^2, \]

Fig. 15. \( -\omega \)-space (a)) and \( k \)-space (b)) pictures of the lasing modes in the: i) \( (1 + 0) \), ii) \( (1 + 1) \), and iii) \( (1 + 2) \)-dimensional cases.
as is usually done in the eikonal approximation of wave optics. This further suggests three relevant physical situations.

(1 + 0)-dimensional optics (fig. 15i)).

Assuming that the laser cavity is a cylinder of length \( L \), with two mirrors of radius \( a \) at the two ends, the cavity resonance spectrum is made of discrete lines separated by \( c/2L \) in frequency, each one corresponding to an integer number of half wavelengths contained in \( L \), plus a crown of quasi-degenerate transverse modes at the same longitudinal wave-number, with their propagation vectors separated from each other by a diffraction angle \( \lambda/a \).

This case corresponds to a gain line narrower than the longitudinal frequency separation (so called free spectral range) and to a Fresnel number

\[
F \equiv \frac{a^2}{\lambda L}
\]

of the order of unity, so that the first off-axis mode already escapes out of the mirror. Intuitively, \( F \) is the ratio between the geometric angle \( a/L \) of view of one mirror from the other and the diffractive angle \( \lambda/a \).

In such a case there is only a time dependence and no space derivatives, that is

\[
\Box^2 \rightarrow 2i\omega d_t .
\]

The resulting ODE replacing the wave PDE has to be coupled to the matter equations giving the evolution of \( P \). For a cavity mode resonant with the control frequency of an atomic transition in a set of equal, two-level atoms, we obtain the so-called Maxwell-Bloch equations

\[
\begin{align*}
\dot{E} &= -\gamma_E E + gP , \\
\dot{P} &= \gamma_\perp P - gEN , \\
\dot{N} &= -\gamma_\parallel N - 2gEP + A ,
\end{align*}
\]

where \( N \) is the population inversion, \( \gamma_E, \gamma_\perp \) and \( \gamma_\parallel \) are the loss rates of \( E \), \( P \) and \( N \) respectively, \( g \) is the field-matter coupling constant and \( A \) is the pump rate.

Equations (9) are isomorphic to Lorenz equations for a model of convective fluid instability. Being three non-linear equations, they provide the minimal conditions for deterministic chaos. However, time scale considerations can rule out some of the three dynamical variables, yielding a dissipative dynamics with only one variable (fixed-point attractor) or two variables (limit cycle attractor). Only when the three damping rates are of the same order of magnitude, we do have a three-equation dynamics and hence the possibility of a chaotic motion (strange attractor). The above three cases have been classified as class \( A, B \) and \( C \) lasers, respectively.

A comprehensive review of experiment and theory for these single-domain, \((1 + 0)\)-dimensional systems is given in ref. [44], covering the period 1982-1987 over which these space-invariant instabilities have been studied.
(1 + 1)-dimensional optics (fig. 15ii)).

Here the cavity is thin enough to reject off-axis modes, but the gain line is wide enough to overlap many longitudinal modes. The superposition of many longitudinal modes means that one must retain the z-gradient. Thus the wave equation reduces to

\[ (\partial_t + c\partial_z)E = GP, \]

where \( G \) is a scaled coupling constant.

Having a PDE, any mode expansion with reasonable wave number cut-offs provides a large number of coupled ODEs, thus it is immaterial whether \( P \) and \( N \) are adiabatically eliminated. Anyway, we have enough equations to see space-time chaos.

(1 + 2)-dimensional optics (fig. 15iii)).

Suppose the gain line allows for one longitudinal mode, but the Fresnel number is sufficiently high to allow for many transverse modes. Then eq. (5) can be reduced to

\[ \left( \partial_t + c\partial_z - \frac{ic}{2k} \nabla_\perp^2 \right)E = i \frac{\omega}{2\varepsilon_0} P, \quad (\nabla_\perp^2 = \partial_x^2 + \partial_y^2), \]

which has to be coupled with the material equation for \( P \). In this paper, rather than dealing with a collection of identical two-level atoms, which provides equations as the second and third eq. (9), we refer to a weakly excited medium so that

\[ P/\varepsilon_0 = \chi E. \]

The time and space scales are so widely different from those of the optical carrier wave, that it is justified to take the \( P \) amplitude as the product of two separate amplitudes \( E \) and \( \chi \), \( E \) obeying the slow eq. (11) and obeying a reaction-diffusion equation as

\[ \left( \partial_t + \frac{1}{\tau} - D \nabla^2 \right)\chi = f(E). \]

5.2. Passive optical systems. – We consider two fundamental schemes in a ring cavity (fig. 16). In fig. 16a), a thin slice of passive medium is crossed by an impinging optical beam, which is then re-applied after propagation through a ring cavity, in a feedback configuration. An alternative set-up, as originally introduced by Firth (10), consists (fig. 16b)) in the in-line superposition of a forward field \( F \) and a back-reflected \( B \) propagated over a free space of length \( L \). We consider a Kerr medium with

\[ f(E) = \alpha |E|^2. \]

By «thin slice» we mean that its thickness is \( l \ll \sqrt{D\tau} \) so that the \( \nabla^2 \) in eq. (13) can be reduced to a two-dimensional \( \nabla_\perp^2 \). The relaxation time of a Kerr medium as a liquid crystal is \( \tau \approx 0.1 \text{–} 1 \text{ s}, \text{ i.e.} \) many orders of magnitude larger than the transit time of light over the cavity length \( L \). We take further \( L \gg l \).
In view of this, eq. (11) can be split into

\[(11') \quad \left( \partial_z - \frac{i}{2k} \nabla_z^2 \right) E = 0 \quad \text{out the medium}, \]

\[(11'') \quad \partial_z E = i\chi E \left( \alpha = \frac{\omega}{2\varepsilon_0 c} \right) \quad \text{inside the medium}. \]

Since we are dealing with the linear case, it is convenient to Fourier transform the transverse dependence ($r_1 \rightarrow q$) in order to evaluate the local field $E$, to be introduced into eq. (10), in terms of standard relations of a feedback amplifier. Integrating eq. (11'), the open loop gain is given by

\[(15) \quad A = \exp \left[ ib\chi \right] \equiv 1 + ib\chi, \]

where $b = al$ ($l =$ thickness of the Kerr slab) and the Kerr medium has been taken as optically thin. Across the feedback path, by $z$-integration of the $q$-transform of eq. (11') ($\nabla_z^2 \rightarrow -q^2$) we have a transfer function (taking unity reflectivity for all mirrors)

\[(16) \quad \beta = \exp \left[ -iL \frac{q^2}{2k} \right]. \]

The field to be applied to the source term eq. (14) of the medium evolution is then given in terms of the external field $E_{in}$ as

\[E_{in} \left( 1 + \beta \frac{A}{1 - \beta A} \right) \equiv E_{in} \left( 1 + i\beta \chi \exp \left[ -iL \frac{q^2}{2k} \right] \right) \]

whose modulus squared is, at first order ($I_{in} = E_{in}^2$)

\[(17) \quad |E|^2 = I_{in} \left( 1 + 2b\chi \sin \frac{L}{2k} \frac{q^2}{2} \right). \]
Introducing this into the $q$-transform of eq. (13), it results an average, $q = 0$, component

$$
\chi(q = 0) = \tau a I_m.
$$

For $q \neq 0$, taking a time dependence as $\chi_q = \exp[\lambda t]$, it is easily found

$$
\lambda = -\frac{1}{\tau} - Dq^2 + 2b I_m \sin \frac{L}{2k} q^2,
$$

which yields a threshold profile $I_1(q)$ of $I_m$ (take $\lambda = 0$) as in fig. 17a, and bands of allowed oscillations (for $\lambda > 0$ at fixed input $\bar{I}$) as in fig. 17b).

Similar results were originally derived by Firth [10] with the scheme of fig. 16b) under the assumption that in the local intensity the interference term can be cancelled, that is

$$
I = |F + B|^2 \approx |F|^2 + |B|^2
$$

since the diffusion length $\sqrt{D\tau}$ is much larger than the optical wavelength $\lambda$.

5.3. Active optical morphogenesis. – We consider a photorefractive crystal, pumped by an argon laser beam and emitting within a ring cavity.

The geometry of the experiment is summarized in fig. 18. For a fixed pump intensity and angle $\theta$, the dynamics is ruled by two control parameters, the d.c. field $E_{d.c.}$ applied across the crystal, in a direction parallel to the grating wavevector, and the aperture $a$ of the pupil of the cavity, which controls the Fresnel number $F$, that is, the number $N = F^2$ of cavity modes. Here we call cavity modes the free field configurations allowed by the boundary conditions.

Increase of the extensive parameter $F$, for fixed pump and $E_{d.c.}$, induces a series of bifurcations described in sect. 4. For low $F$ a single mode per time appears, but a given number $N$ of modes, limited by the relation $N \leq F^2$, competes sequentially. This means that each of the $N$ modes survives for a given time, and then it is replaced by another one and so on, in a periodic or chaotic alternation (PA and CA). Away from the narrow switching time intervals, the amount of mode mixing is negligible.

As $F$ increases further, pure cavity modes no longer alternate, but they coexist at the same time with little mutual correlation, as shown by the shrinking of the
correlation length. The many bright spots which are observed on the wavefront or, equivalently, the intermingled dark spots giving rise to phase singularities, are no longer lying in a regular pattern. In fact what we now observe is an almost stochastic pattern.

The model here presented accounts for all the relevant physical aspects without recurring to unnecessary microscopic assumptions. The requirement is that all the relevant parameters should be characterized by the experimental set-up shown in fig. 18 plus the addition of simple optical measurement of bulk properties done on the crystal without cavity [23].

The two-wave mixing gives rise to a grating which breaks the cylindrical symmetry of the cavity. The grating spacing is

\[
\Lambda = \frac{2\pi}{K} = \frac{2\pi}{k_0 \theta},
\]

\(k_0\) being the wave number of the pump, and \(\theta \ll 1\) the angle between the two waves, which in the experiment [17] is optimized at \(\theta = 1/60\ \text{rad}\).

We introduce an \(x, y, z\) coordinate system with \(z\) along the cavity axis, \((x, z)\) being the plane which contains the propagation directions of pump and signal fields \(E_p\) and \(E_s\), and with the origin at the intersection of the input face of the crystal with the cavity axis. The polarization \(P\) depends on the total field \(\vec{E} = E_p + E_s\).

The susceptibility field obeys the equation

\[
\left( \partial_t + \frac{1}{\tau} - D \nabla^2 \right) \chi = f(\vec{E}),
\]

which arises from the following considerations. The two-wave mixing gives a space charge field consisting of a dipole distribution created by the offset of the free electrons with respect to the fixed donors in the BSO material. The collective wave \(\chi\) decays locally with a lifetime \(\tau\) and diffuses at a rate \(D\). Direct measurements
on the crystal used provide a value \( \tau = 1 \text{s} \), while values of \( D \) reported in the
literature are spread in the range \( 10^{-6} - 10^{-10} \text{m}^2/\text{s} \).

We first isolate a fast-varying part in \( z \) and \( t \) from the slow space-time dependence
which is relevant for the dynamics, by writing

\[
\tilde{E} = E_p \exp[-i(k_0 r - \omega_0 t)] + E(r_\perp, z, t) \exp[-i(kz - \omega t)].
\]

This way, we have extracted from the cavity field a plane wave along the cavity axis
\( k \) along \( z \), thus \( E \) includes the full dependence on \( r_\perp \) plus the residual slow
dependence on \( z \) and \( t \).

The applied d.c. field in the \( x \)-direction induces a space charge drift, with a speed
\( v_d \), thus, from eq. (2), the scattering grating provides a frequency offset along \( x \)
given by

\[
\Omega = v_d k_0 \theta.
\]

Consequently, the cavity field has a frequency \( \omega \) detuned with respect to the pump
frequency \( \omega_0 \) by

\[
\omega = \omega_0 - \Omega.
\]

We neglect this detuning for the following reason. In terms of the transverse wave
number \( q (|q_{\text{max}}| = 100 \text{cm}^{-1}, \text{limited by the diffusive cut-off } q_c^{-1} = 100 \mu\text{m} \text{ of the}
\text{crystal}), \text{propagation introduces a diffractive dephasing } \delta \varphi_1 = q^2 (L/2k) = O(1), \text{since}
L = 2 \cdot 10^5 \text{cm}.

Now, since the bandwidth of the active material is very narrow \( (\Delta \nu \equiv 1/\tau \equiv 1 \text{s}^{-1}) \)
all oscillating modes are pulled within this bandwidth, so that the mutual dephasing
between adjacent modes due to detuning is of the order of \( \delta \varphi_2 \equiv \Delta \nu L/c \equiv 10^{-8} \text{ thus}
\text{fully negligible in comparison to } \delta \varphi_1.

In the Fourier expansion of the transverse dependence \( r_\perp \rightarrow k \), only the two
components

\[
\chi_0 \equiv \chi(k = 0)
\]

and

\[
\chi \equiv \chi(-K = k - k_0),
\]

multiplied, respectively, by the cavity and pump fields yield phase-matched con-
tributions \( P = \varepsilon_0 \chi \tilde{E}. \text{ Hence the equation for the signal field reduces to}

\[
(23) \quad \left( \partial_t + c \partial_z - \frac{ic}{2k} \nabla^2_\perp \right) E = -\frac{i \omega}{2} \left( \chi_0 E + \chi E_p \right) \text{ rect}(z/l) =
\]

\[
= \left( -\frac{i \omega}{2} \chi E_p - IE \right) \text{ rect}(z/l),
\]

where \( \text{rect}(z/l) = 1 \) only within the crystal thickness \( 0 < z < l \), and is zero elsewhere.
In writing the last line we have taken into account only the imaginary part of \( \chi_0 =
= \chi_0' + i \chi_0'' \), through \( I' = -(\omega/2) \chi_0'' \). This corresponds to a transmission loss of the
\text{crystal in the absence of pump which persists unmodified even with pump. The real}
\text{part of } \chi_0 \text{ induces a renormalization of } c \text{ which is irrelevant for the field equation,
since it has to be weighted by the cavity filling factor \( l/L \ll 1 \). Thus the material eq. (21) provides the evolution for the envelope of the susceptibility

\[
\chi = \tilde{\chi}(r_\perp, t) \exp \left[ i(Kx - \Omega t) \right].
\]

By the usual argument of separation of fast and slow variables in the longitudinal dependence, matching of the phases which appear in the exponentials of eq. (24) and (22) accounts for the «fast» space variations on a scale

\[
\frac{2\pi}{K} \equiv \Lambda = \frac{\lambda}{\theta} \approx 30 \mu m, \quad (\theta \approx 1/60, \lambda \approx 0.5 \mu m).
\]

We can rewrite the equation for the slow \( \chi \) envelope as

\[
\left( \partial_t + \frac{1}{\tau} - D \nabla^2 \right) \tilde{\chi} = r \frac{E_p^* E}{|E_p|^2} \left( 1 - \frac{|E|^2}{|E_p|^2} \right),
\]

where the source term is derived from the standard model of photorefractive media, with the assumption \( |E| \ll |E_p| \) and \( r \) is the complex photorefractive coefficient. For the present purposes it is sufficient to keep only the linear part of this equation, thus we rewrite it as

\[
(25') \left( \partial_t + \frac{1}{\tau} - D \nabla^2 \right) \tilde{\chi} = AE.
\]

The active system is somewhat different from the passive one. The main differences are the following:

i) within the BSO crystals, there are strong losses \( l' \approx 1.9 \text{ cm}^{-1} \), leading to a transmission of only 15% in one passage. Within the thickness \( l \) of BSO eq. (23) reduces to

\[
\partial_z E = -i\chi \tilde{\chi} - l' E \quad \left( \chi = \frac{i\omega}{2c} E_p \right),
\]

ii) the diffusively thin approximation is not appropriate, since \( l = 1 \text{ cm} \) (thinner crystals do not have enough gain to yield oscillations) against a diffusive length \( l_c = \sqrt{D\tau} \approx 100 \mu m \).

For the moment, let us consider a diffusively thin medium \( l \ll \sqrt{D\tau} \) so that \( \tilde{\chi} \) has no \( z \)-dependence. In such a case, eq. (26) can be easily integrated to provide

\[
(27) \quad E(1) = E(0) \exp \left[ -l' l \right] - \frac{i\chi}{l} \tilde{\chi}(1 - \exp \left[ -l' l \right]).
\]

In the limit of an optically thick medium \( (l' l) \gg 1 \) this reduces to

\[
(28) \quad E(r_\perp, l) = -\frac{i\chi}{l} \tilde{\chi}(r_\perp).
\]

Thus, the output field carries no memory of the input field, but it follows the \( \chi \)-dependence.

In the \( q \)-transform space, the free propagation of the output field across the length
$L \gg l$ is given by the solution of the $t$-independent $q$-transform of the source-free eq. (23), which is

$$\left( \partial_z + \frac{i}{2k} q^2 \right) E_q = 0$$

that is, the field to be introduced into the material equation is

$$E_q = \exp \left[ -iL \frac{q^2}{2k} \right] E_q(l) = \exp \left[ -iL \frac{q^2}{2k} \right] \tilde{\chi}_q.$$  

In the diffusively thin assumption, in the material eq. (25) we have $\nabla^2 \rightarrow \nabla^2_1 \rightarrow -q^2$ thus we must solve the eigenvalue equation for $\tilde{\chi}_q = \exp[\lambda q t]$, which is

$$\lambda + \frac{1}{\tau} + D q^2 = \exp \left[ -iL \frac{q^2}{2k} \right],$$

that is, when we consider the real part of $\lambda$ we have a condition similar to eq. (19) and hence instability bands as in fig. 17.

A more realistic approach, considering the correct optical and diffusive thickness including a $z$-integration, has been carried out [45] and it has provided a diagram which looks like that of Turing morphogenesis, without allowed secondary bands (fig. 19).

5.4. Normal form approaches. Beyond the onset of separate, independent modes, selected by the $\lambda - q$ diagram, non-linearities induce the mode competition, resulting in periodic or chaotic alternation (PA and CA) as shown in sect. 4.

The normal form equations are the simplest variety of non-linear model. They consist in the simultaneous consideration of a discrete (usually small) number of mode amplitudes, with their linear evolution as it results from the analysis sketched above, but including the leading mode-mode non-linear coupling compatible with the symmetry requirements of the physical problem. In the $N$-dimensional phase space spanned by the $N$ mode amplitudes, any one of the $N$ fixed points may be of saddle type with some contracting and some expanding directions. This gives rise to heteroclinic cycles arriving close to the various fixed points. This approach, given in
all its generality in ref. [15], was initially exploited in a stationary problem, that is, the successive onset of different configurations as a control parameter is varied [46,47]. We have applied the method to model the dynamical trajectories at fixed-control parameter, reaching a satisfactory agreement between the model trajectories and those resulting from the experiment [16].

6. – Conclusion.

It is rather difficult to put an end to a review of work in progress. One area which has been heavily overlooked is that of morphogenesis in lasers. The main advantage of the systems here considered (Kerr and photorefractive media) is that their time scales are so slow, that many transient features can be followed in real time and compared with the models, whereas in the laser case only time-integrated patterns are available thus far, and a breakthrough is necessary in order to reach a satisfactory match between models and experiment.

I have not mentioned possible technical applications as optical parallel computers, since in this regard there are still vague speculations rather than working schemes.

The guideline followed in this review has been the comparison between a reaction-diffusion system, which rules Turing chemical instabilities as well as all classes of biological and physiological models, and diffraction-diffusion systems, where coherent phase propagation may lead to a large set of unstable bands.

We have seen that since the diffusion within the coupled medium introduces a cut-off $q_c$, only one band may be within the allowed bandwidth as we account for damping in the medium. The question is important in some recent diffraction experiments in quantum mechanics, where the Schrödinger wave function of a cooled atomic beam is diffracted by a stationary laser wave [48].

I have recently considered a «Gedanken-experiment» [49] whereby, after interaction with the laser wave, the atomic beam undergoes free propagation and then is reflected back, returning to the laser region. Free propagation transforms the phase modulation, born in the interaction, into an amplitude modulation, and thus in a perturbation of the probability density $|\psi|^2$ of the back atomic beam.

If the change in $|\psi|^2$ perturbs the laser excitation, we have a feedback scheme analogous to those considered in sect. 4. From whence the possibility of morphogenesis in quantum mechanics, and that of a phase damping in a kind of «high Fresnel number» limit. Phase damping means that, whenever a microscopic (diffractive) dynamical system is coupled to a macroscopic (diffusive) one, as the confinement space (integration boundaries) enlarges, superposition disappears and the description reduces to classical probabilities. All this is matter of future speculations.

REFERENCES


[8] Self-organization has been used to denote in general the onset of macroscopic spatial structures in steady state; see V. I. Krinsky (Editor): *Self-organization, Auto-waves and Structures far From Equilibrium* (Springer-Verlag, Berlin, 1984). Whenever boundaries impose their own symmetry constraints, the resulting patterns are organized from outside, hence I call them «hetero-organized». I propose to call strictly «self-organized» only those cases where the organization emerges from the intrinsic properties of the medium (scales expressed in terms of the coefficients of the PDE).


[27] We have focused on delays much longer than the characteristics time of the dynamics (i.e. the period in the case of regular oscillations, or the correlation time in the case of stochastic oscillations) showing that the corresponding high-dimensional dynamical system can be represented in two dimensions (a pseudo-space-time representation: F. T. Arecchi, G. Giacomelli, A. Lapucci and R. Meucci: *Phys. Rev. A*, 45, 4225 (1992)). A relevant modelling can be achieved by Coupled Map Lattices (S. Lefri, G. Giacomelli, A. Politi and F. T. Arecchi: *Physica D*, 70, 235 (1994)). The two-dimensional representation provides explicit evidence of phase defects (G. Giacomelli, R. Meucci, A. Politi and
F. T. ARECCHI: Defects and space-like properties of delayed dynamical systems, submitted for publication.


[33] See the first of ref. [11].


