Chaos, Complexity and Morphogenesis: Optical-Pattern Formation and Recognition(*)

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Summary. — In 1952 A. Turing introduced the concept of chemical morphogenesis. A medium with at least two interacting and diffusing components (activator and inhibitor) can be subjected to spontaneous pattern formation, with a scale length independent of the boundary conditions, and thus maintained even in the infinite volume limit. This is at variance with pattern formation in fluids (as, e.g., Rayleigh-Bénard and Faraday instabilities) where the size is imposed by the boundary geometry. In non-linear optics, patterns emerge from the coupling of a diffractive equation describing electromagnetic propagation with a diffusion equation describing the local modification of the polarizability in a medium. As we adjust an extensive parameter (the so-called Fresnel number $F$) corresponding to the optical aspect ratio, we observe a transition from a regime dominated by the boundary constraints to a Turing-type regime dominated by the bulk parameter. This is equivalent to saying that the prominent role is due to the diffractive equation in one case and to the diffusive one in the other. Morphogenesis for low $F$ arises from the non-linear competition among a small number of degrees of freedom, giving rise to a space-uniform excitation with a low-dimensional dynamics. This gives rise to the different scenarios of chaos. Their properties have been explored in the past decade. In the large-$F$ case, the space-time instabilities rapidly evolve toward complex patterns, not reducible to a few indicators. In the case of two-dimensional fields, global characterization is achieved via the statistics of the topological defects.

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1. – Introduction.

In a conference on complex liquids, it is appropriate to survey what has been done in the area of optical patterns. Reaction-diffusion processes give rise to chemical

patterns (space-dependent distribution of concentrations of different species) by supercritical bifurcations. Indeed Turing instabilities correspond to the onset of patterns by a continuous transition (as a second-order transition in equilibrium media) [1,2]. In excitable media, where multiple stable branches can coexist, subcritical bifurcations connect stable branches, giving rise to jumps as observed in neural transmission and cardiac excitation [3].

In fluids, convective instabilities induced by thermal, concentration or pressure gradients give rise to varieties of phenomena eventually leading to turbulence [4,5]. In optics, there is one more fact to consider, that is, the particle-particle interaction is not just short range as with collisions, or van der Waals forces etc., but it can be propagated over very large distances by the wave character of the field propagation in free space. Wave phenomena play already a role in a fluid pattern, as e.g., Faraday instabilities. However, in fluids the non-linear medium must have a continuity whereas in optics large empty spaces can be put between two interacting domains.

In view of this, we can speak of optical crystals and optical turbulence dynamical patterns detected in empty space, rather than corresponding to a repositioning of material particles.

A general review on pattern formation has recently appeared [6]. More specific surveys of optical-pattern formation are also available [7]. We refer to them for an exhaustive collection of references. Here, we limit to pick up some problems which have a conceptual relevance for this conference.


In this section we discuss the interaction of an optical field in feedback configuration with a passive medium where the susceptibility depends on the field intensity (Kerr medium).

The system consists of a liquid-crystal light valve (LCLV) in a ring optical cavity (fig. 1). The LCLV is a mirror sandwiched between a nematic-liquid-crystal (LC) layer and a photoconductive layer. An AC sinusoidal voltage is applied across the photoconductive and the LC layers. In the usual operating conditions, the r.m.s. voltage amplitude is about 20 V and its frequency is 20 kHz. The liquid-crystal director (axis of the aligned LC molecules) is vertical as well as the input beam polarization. In the absence of light, most of the voltage drops across the photoconductive layer. The rest, applied to LC, tends to orient the molecules longitudinally, thus changing the refractive index. In the presence of illumination the photoconductive layer has a smaller voltage drop, the voltage applied to LC increases and further reorientation of the molecules induces a phase shift \( \Delta \phi \) of the reflected beam proportional to the intensity \( I \) of the beam incident on the back of the LCLV. In this way the system acts as a Kerr-like medium. The non-linearity is defocusing because the LC molecules tend to align longitudinally, thus decreasing the refractive index along the propagation direction.

The advantage of using the LCLV instead of a simple layer of LC is that the non-linear phase shift requires a low input intensity (around 5mW/cm²) and can be controlled by changing the applied voltage.

21. Optical crystals and quasi-crystals. As shown in sect. 4, at the linear stage a circle of critical transverse wave vectors \( q \) become simultaneously unstable. Close to threshold, the pattern is determined by a non-linear interaction among wave vectors on this circle which has a radius \( q_1 = \sqrt{2k_0} / L \) for a focusing medium and \( q_{11} = \sqrt{2k_0} / L \) for a defocusing medium, \( k_0 = 2\pi / \lambda \) being the optical wave number, and \( L \) being the free propagation length.

The free end of the fiber bundle is mounted on a rotation stage which allows for a continuous angular positioning over a full 360° range with a high resolution (readout to 0.2°). This way, the feedback image arriving at the photoconductive layer of the LCLV can be rotated of any angle \( \delta \). For \( \delta = 0 \) only hexagons are stable. On the other hand, for \( \delta = \pi \) only rolls are stable.

As the rotation angle is exactly commensurate to \( 2\pi \), that is, when \( \delta = 2\pi / N \) with \( N = 2, 3, 4, \ldots \), a pattern with an \( N \)-fold symmetry develops. Reference [8] is a systematic experimental investigation of these symmetries, with a quantitative modeling.

In fig. 2 we report near-field patterns, obtained for various rotation angles \( \delta = 2\pi / N \), where \( N \) is changed from 2 to 9. Since the critical wavelength is of the order of 1 mm and the LCLV has a diameter of 3 cm, a central region with negligible boundary influences has been observed up to \( N \) of the order of 20. Looking at the edge, it can be seen that away from the centre the system tends to stabilize rolls for \( N \) even and hexagons for \( N \) odd. In fact, the feedback rotation constraint is more efficient close to the centre, whereas at the edge the patterns recover the two basic symmetries corresponding to \( \delta = 0 \) (\( N \) odd) and \( \delta = \pi \) (\( N \) even).
3. Active optics: photorefractive oscillators.

31. PA, CA, STC (periodic and chaotic alternation, space-time chaos) [9]. We seed a ring cavity with a photorefractive gain medium pumped by an argon laser and study the temporal and spatial features of the generated field. By varying the size of a cavity aperture, it is possible to control the number of transverse modes which can oscillate. The experimental set-up consists of a ring cavity with gain. The gain medium is a photorefractive (5 x 5 x 10) nm² BSO (bismuth silicon oxide) crystal to which a d.c. electric field is applied. The crystal is pumped by a c.w. argon laser with an intensity around 1 mW/cm².
The Fresnel number of the cavity is controlled by a variable aperture. The Fresnel number gives the maximum order of the transverse modes that can oscillate. $F$ can be varied in the range from 0 to approximately 100. Correspondingly, the total number of transverse modes allowed by diffraction scales as $F^2$.

Transverse intensity patterns corresponding to increasing Fresnel numbers $F$ are shown in fig. 4 together with fluctuations around the local average. The low-$F$ limit ($F \leq 4$) corresponds to a time alternation among pure cavity modes (fig. 4a), yielding a spatial correlation length $\xi$ of the order of the transverse size $D$ of the beam. For high $F$ ($F = 15$), on the contrary, the signal appears as a speckle-like pattern irregularly evolving in space and time (fig. 4c), with a short correlation length ($\xi/D < 0.1$). The transition between these two limits is characterized by a continuous variation of the ratio $\xi/D$. A generic intermediate situation is shown in fig. 4b. An example of alternating pure-mode configurations is given in fig. 5.

In order to study the time behaviour of the system we perform a local measurement of intensity vs. time by placing an optical fiber in an arbitrary point on the wave front, so that each mode is encoded by a local intensity level. Identifying each mode with its azimuthal quantum number (defined as half the number of nodes along the circumference), we have modes $7, 6, 5, 4, 3, 2, 1, 0$ alternating on a time scale of seconds, that is, of the order of the dielectric relaxation time of BSO (fig. 6).

The alternation is peculiar of low Fresnel numbers. It is observed down to the minimum $F$ for which the cavity can oscillate. While for large apertures the field pattern may be expanded in a large number of cavity modes, for small apertures the field at any time is made of a single mode, however a small number of modes (from two to about ten) alternate in time. Thus, the alternation phenomenon consists of an ordered sequence of quasi-stationary modes. Depending on some control parameter, the persistence time of each mode is either regular.

**Fig. 4.** Photorefractive oscillator in a ring cavity (ref.[9]): intensity distribution of the wave front (left) and spatial autocorrelation function (right) for increasing Fresnel number. a) $F = 1$, one single mode at a time is present, ratio between coherence length $\xi$ and frame size $D$ is $\xi/D = 1$; b) $F = 7$, $\xi/D = 0.2$; c) $F = 10$, $\xi/D = 0.06$.

**Fig. 5.** Photorefractive oscillator: intensity patterns of the pure modes in their order of consecutive appearance in a cycle of periodic alternation at $F = 6$ (from ref.[9]).
local chaotic signal is not confined in a low-dimensional space and we speak of STC.

32. Phase singularities. A phase singularity is a point around which the circulation of the phase gradient is a multiple of $\pm 2\pi$. We call topological charge the multiplicity number. In the case of the wave equation only $\pm 1$ charges are stable.

The nature of these singularities is determined by the fact that $E$ is a smooth single valued function of its variables. Single valuedness implies that during a circuit $C$ in space-time $\phi$ may change by $2m\pi$, where $m$ is an integer. Suppose $m$ is not zero and let $C$ be shrunk to a very small loop in such a way that $m$ does not change. Then $C$ encloses a singularity because $\phi$ is varying infinitely fast. The smoothness of $E$ now implies that this can happen only where $E = 0$, i.e., where $\phi$ is indeterminate. Since the vanishing of $E$ requires two conditions ($\Re E = \Im E = 0$) these phase singularities are lines in space or points in the plane (fig. 7a)). Sometimes we call the phase singularities as wave front dislocations or, by analogy with fluid dynamics, vortices.

Let us consider a random field $E(x, y)$. If it is formed by the interference of a large number of independent components, then $\Re E$ and $\Im E$ are two independent random functions with Gaussian statistics.

The zeroes of the function $\Re E(x, y)$ determine a number of curves in the $(x, y)$-plane, see fig. 7b). There is another set of curves corresponding to $\Im E(x, y) = 0$, and now the intersections of curves of one family with those of the other give discrete points in which $|E(x, y)| = 0$. If we consider the problem of the propagation of such zeroes along the direction in accordance with the wave equation, the discrete points in the $(x, y)$-plane are converted into lines. It is clear that in general these lines do not intersect in three-dimensional space. Moreover, a given line neither appears singly at some plane $z = \text{const}$, nor can it disappear singly. Zeroes in the pattern must appear or be annihilated in pairs.

Up to now we spoke of a random speckle pattern. It is clear however that the topological dimensional arguments do not depend on the nature of the interfering fields. The only difference is that for regular fields, zeroes may not occur at all.

![Figure 6](image1.png)

Fig. 6. Periodic alternation in a photorefractive oscillator. Time record of local intensity at $P = 5$ (from ref. [9]).

![Figure 7](image2.png)

Fig. 7. a) Pair of phase singularities, or wave front dislocations, of opposite topological charge. b) System of dislocations in a random field. Solid lines $\Re E = 0$, dashed lines $\Im E = 0$. 

(periodic alternation: PA) or irregular (chaotic alternation: CA). Away from the narrow switching time intervals, the amount of mode mixing is negligible.

A phenomenon similar to CA, called chaotic itinerancy, was introduced by several authors in dealing with numerical solutions of different classes of models as globally coupled iteration maps[10]. In fact, itinerancy includes erratic jumps among the available quasi-stationary states, where CA keeps the sequence ordering.

Increasing the value of the control parameter, we enter a new regime, called spatio-temporal chaos (STC) where a large number of modes coexist. This regime is characterized by a short correlation range defined as follows. Suppose we have a generic field $u(r, t)$, ruled by a partial differential equation including non-linear and gradient terms. Let us consider the deviations away from the local time average

\[ \delta u(r, t) = u(r, t) - \langle u(r, t) \rangle, \]

where $\langle \ldots \rangle$ denotes time average. Under very broad assumptions, we can take the leading part of the correlation function as an exponential, that is

\[ C(r, r') = \langle \delta u(r, t) \delta u(r', t) \rangle \equiv \exp[-|r - r'|/\xi]. \]

Whenever the correlation length $\xi$ is larger than the wave front size $a(\xi > a)$ we have low-dimensional chaos, that is, even though the system can be chaotic in time, it is coherent in space (single mode, in a suitable mode expansion). The corresponding chaotic attractor is low dimensional. In the limit $\xi \ll a$, a
The difference of \( n_+ \) and \( n_- \), the numbers of zeroes with positive and negative charges, is conserved in the process of propagation. On the average in a cross-section of a speckle field \( n_+ = n_- \) since the beam is statistically homogeneous.

The dislocation density coincides with the number of speckles per unit area. This was verified in a series of experiments\([11]\), where the speckle field was obtained by transmitting a laser beam through a distorting phase plate. The structure of the density \( N \) scales like \( a^2 \) (\( a \) is the diaphragm diameter on the phase plate) and thus linearly in \( F \); thus the total number of dislocations \( N a^3 \) scales like \( F^2 \).

In non-linear optics we have recently shown experimental evidence of phase singularities\([12]\). Their positions and the scaling of their separations, number and charges with the Fresnel number allow a classification of patterns.

At variance with the material waves which are easily visualized in terms of matter displacements, in the case of an optical field a phase measurement requires heterodyning against an external reference. Phase information is extracted by beating the signal with a reference beam onto a CCD videocamera.

By a suitable algorithm, we reconstruct the instantaneous surfaces of phase as shown in fig. 8 where the phase surface of a doughnut mode is a helix of pitch \( 2\pi \) around the core (vortex).

When more than one vortex is present, in order to resolve and count each vortex, we tilt the reference beam so that the video signal is now given by

\[
I(x, y) = A^2 + B^2 + 2AB \cos(Kx + \varphi(x, y)),
\]

where \( A \) and \( B \) are the amplitude of reference and signal field, \( K \) is the fringe frequency due to tilting, \( x \) is the coordinate orthogonal to the fringes and \( \varphi \) is the local phase. This way, a phase singularity appears as a dislocation, and the topological charge is visually evaluated.

We digitize the fringe system and count those defects separated by at least one fringe, in the region where fringes can be resolved. Figure 9a) shows a configuration with an overall unbalance in the topological charge. A heuristic explanation of fig. 9a) is that, for small \( F \), the dynamics is strongly boundary dependent. Consequently we expect that an increase of \( F \) should eventually yield the thermodynamic limit of paired charges. This is indeed the case shown in fig. 9b), which refers to a high \( F \) and where the charge unbalance \( U = |n_+ - n_-| \) has become very small compared to the total number \( N = n_+ + n_- \) of charges.

By averaging over a large number of frames for each \( F \), we can find \( \langle N \rangle \), the mean number of singularities per frame, and \( \langle D \rangle \), the mean nearest-neighbour distance. These are plotted as a function of \( F \) in fig. 10. Notice that, for low \( F \),
excess is large. For increasing $F$, $U_N$ decreases as a power law with exponent close to $-1.5$.

The spatial disordered of singularity positions is associated with the passage to STC. Since a phase singularity must be associated with a zero crossing of real and imaginary parts of the field, it follows that all intensity zeroes are singularities. But the diffractive treatment of optical cavities shows that the number $N$ of intensity zeroes for the highest allowed mode scales as the square of $F$ ($N \sim F^2$). On the other hand, if $a$ is the pupil aperture of the optical system, and $D$ is the average interdefect separation, we expect $N \sim a^2/D^2$, and, since $a^2 = F$ then $D \sim F^{-2}$. Such scaling laws are verified in fig. 10. We can justify also the $F$-dependence of $U_N$. Assume that unpaired defects are mainly created at the boundary, while in the bulk pairs with compensated charge are created and destroyed. Then the total number $N_c$ of boundary defects in the perimetal region of area $a$. $D$ scales as $N_c = aD/D^2 = a/(D) = F^1$, and the corresponding unbalance is $U = \sqrt{N_c} \sim F^{0.5}$. Hence the normalized unbalance scales as $U_n = F^{0.5-2} = F^{-1.5}$, in accord with the experiment.

33. Transition from boundary- to bulk-controlled regimes. – As we increase the Fresnel number, we see the transition from patterns dominated by the geometric parameters to dissipative patterns whose scale length is imposed by the bulk properties of the medium.

A fundamental geometric parameter is the spot size of the central mode which is constrained by the quasi-confocal configuration. The spot size of the central mode is
given by

$$w_0 = \sqrt{\frac{\lambda L}{\pi}}.$$  

Provided the mirror size $a$ is larger than $w_0$, that is the Fresnel number $F = a^2/\lambda L$ is larger than 1, the cavity can house higher-order modes, made of regular arrangements of bright spots (peaks of Gauss-Laguerre functions in cylindrical geometry) separated by

$$D \equiv \frac{\lambda w_0}{\sqrt{F}}.$$  

Since the overall spot size of a transverse mode of order $n$ scales as $\sqrt{n} w_0$, it is clear that $n = F$ is the largest-order mode compatible with the boundary conditions (filling all the aperture area). Notice that patterns built by superposition of Gauss-Laguerre functions have in general an average separation $\langle D \rangle$ of zero intensity points approximately equal to the average separation $D$ of bright peaks [14].

Extending the range of the explored $F$ values, the plot of $\langle D \rangle$ vs. $F$ (fig. 10(a)) shows that eq. (4) is verified up to a critical value $F_c$, above which $D$ is almost independent of $F$. In a similar way, the total number $N$ of phase singularities scales as $F^2$ or $F$, respectively below and above $F_c$. This transition is evident in fig. 10(b) and its root is the following. $N$ is the ratio of the total wave front area $a^2$, which scales as $F_c$, to the area $\langle D \rangle^2$ containing a single defect, which scales as $F^{-1}$ or $F^3$, respectively below and above $F_c$.

We can understand the transition as follows. Assume that the photorefractive crystal is a collection of uncorrelated optical domains, each one with a transverse size limited by a correlation length $l$, intrinsic to crystal excitations. Then the medium gain has an upper cut-off at a transverse wave number $1/l$, and spatial details are amplified only up to that frequency, that is provided they are bigger than $l$. Thus for a critical $F_c$ such that $D = w_0/\sqrt{F_c} = l_w$ we expect a transition from a boundary to a bulk-dominated regime where the separation of the phase singularities is independent of $F$. This is indeed the case as shown in fig. 10 which yields a value $F_c \approx 11$ corresponding to $l_w = 170 \mu m$, when $w_0 = 600 \mu m$ for $L = 200 \ cm$.

The reduction of the boundary influence is also signalled by the reduction of the topological charge imbalance. Indeed, since a regular field should have a balance between topological charges of different sign, an imbalance means that two phase singularities of opposite sign have been created close to the boundary and only one has remained within the boundary. Therefore, there is a boundary layer of area $a/\langle D \rangle$ containing $N_{1} = a/\langle D \rangle$ singularities. The absolute value of the imbalance is, for statistical reasons, of the order of $\sqrt{N_1}$, and thus the normalized imbalance $U = \sqrt{N_1}/N$. Accounting for the scaling of $\langle D \rangle$ with $F$, it follows that $U$ scales as $F^{-1.8}$ and $F^{-0.67}$, respectively below and above $F_c$.

4. - Modeling optical morphogenesis.

4.1. General features. - Since optical laboratory phenomena are confined in a cavity mainly extended in a $z$-direction (as, e.g., the Fabry-Perot cavity), we expand

$$\Delta^2 e = -\mu e^2 \frac{\partial}{\partial t} p$$

(we simplify derivatives as, e.g., $\partial^2_t$ rather than $\partial^2_j/\partial t^2$, and $p(r, t)$ is the induced polarization), as

$$e(r, t) = E(x, y, z, t) \exp[-i(\omega t - kz)].$$

If the longitudinal variations are mainly accounted for by the plane wave, then we can take the envelope $E$ as slowly varying in $t$ and $z$ with respect to the variation rates $\omega$ and $k$ in the plane-wave exponential. Furthermore we call $P$ the projection of $p$ on the plane wave. By neglecting second-order envelope derivatives it is easy to approximate the operator on $E$ as

$$\Delta^2 \rightarrow 2ik \left( \partial_x + \frac{1}{c} \partial_t \right) + \partial_x^2 + \partial_y^2,$$

as is usually done in the eikonal approximation of wave optics. This further suggests three relevant physical situations.

(1 + 0)-dimensional optics (fig. 12(i)).

Assuming that the laser cavity is a cylinder of length $L$, with two mirrors of radius $a$ at the two ends, the cavity resonance spectrum is made of discrete lines separated by $c/2L$ in frequency, each one corresponding to an integer number of half wavelengths contained in $L$, plus a crown of quasi-degenerate transverse modes at the same longitudinal wave number, with their propagation vectors separated from each other by a diffraction angle $\lambda/\alpha$.

This case corresponds to a gain line narrower than the longitudinal frequency separation (so-called free spectral range) and to a Fresnel number

$$F \approx \frac{a^2}{\lambda L}$$

of the order of unity, so that the first off-axis mode already escapes out of the mirror. Intuitively, $F$ is the ratio between the geometric angle $a/L$ of view of one mirror from the other and the diffractive angle $\lambda/\alpha$.

In such a case there is only a time dependence and no space derivatives, that is

$$\Delta^2 \rightarrow 2i\omega \partial_t.$$

The resulting ODE replacing the wave PDE has to be coupled to the matter equations giving the evolution of $P$. For a cavity mode resonant with the control frequency of an atomic transition in a set of equal, two-level atoms, we obtain the so-called Maxwell-Bloch equations for $E, P$ and the atomic inverted population $N$.

These equations are isomorphic to Lorenz equations for a model of convective fluid instability. Being three non-linear equations, they provide the minimal conditions for deterministic chaos. However, time scale considerations can rule out some of the three dynamical variables, yielding a dissipative dynamics with only one variable
Fig. 12. - $\omega$-space (a) and $k$-space (b) pictures of the lasing modes in the: i) (1 + 0)-, ii) (1 + 1)-, and iii) (1 + 2)-dimensional cases.

(1 + 1)-dimensional optics (fig. 12 ii).

Here the cavity is thin enough to reject off-axis modes, but the gain line is wide enough to overlap many longitudinal modes. The superposition of many longitudinal modes means that one must retain the $z$-gradient. Thus the wave equation

\[ (\partial_t + c \partial_z) E = GP, \]

where $G$ is a scaled coupling constant.

Having a PDE, any mode expansion with reasonable wave number cut-offs provides a large number of coupled ODE's. We do not survey these longitudinal patterns here.

(1 + 2)-dimensional optics (fig. 12 iii).

Suppose the gain line allows for one longitudinal mode, but the Fresnel number is sufficiently high to allow for many transverse modes. Then eq. (5) can be reduced to

\[ \left( \partial_t + c \partial_z - \frac{ic}{2k} \nabla_z ^2 \right) E = i \frac{c_0}{2 \tau_t} P \left( \nabla_z ^2 \sigma ^2 + \sigma ^4 \right), \]

which has to be coupled with the material equation for $P$. We consider a weakly excited medium so that

\[ P/\epsilon_0 = \chi \tilde{E}. \]

The time and space scales are so widely different from those of the optical carrier wave, that it is justified to take the $P$ amplitude as the product of two separate amplitudes $E$ and $\chi$, $E$ obeying eq. (11) and $\chi$ obeying a reaction-diffusion equation as

\[ \left( \partial_t + \frac{1}{\tau} - D \nabla ^2 \right) \chi = f(E). \]

42. Passive optical systems. - We consider two fundamental schemes. In fig. 13a) a thin slice of passive medium is crossed by an impinging optical beam, which is then re-applied after propagation through a ring cavity, in a feedback configuration. An alternative set-up, as originally introduced by Firth[16], consists (fig. 13b)) in the in-line superposition of a forward field $F$ and a back-reflected $B$ propagated over a
free space of length $L$. We consider a Kerr medium with

\begin{equation}
(14) \quad f(E) = \alpha |E|^2.
\end{equation}

By "thin slice" we mean that its thickness is $l \ll \sqrt{\tau}$ so that the $\nabla^2$ in eq. (13) can be reduced to a two-dimensional $\nabla^2_z$. The relaxation time of a Kerr medium as a liquid crystal is $\tau = 0.1-1$ s, i.e. many orders of magnitude larger than the transit time of light over the cavity length $L$. We take further $L \gg l$.

In view of this, eq. (11) can be split into

\begin{equation}
(11') \quad \partial_z - \frac{i}{2k} \nabla^2_z = 0 \quad \text{outside the medium}
\end{equation}

\begin{equation}
(11'') \quad \partial_z E = i\alpha_0 E \left( \frac{\omega}{2\epsilon_0 c} \right) \quad \text{inside the medium}.
\end{equation}

Since we are dealing with the linear case, it is convenient to Fourier transform the transverse dependence ($r \rightarrow q$) in order to evaluate the local field $E_z$ to be introduced into eq. (10), in terms of standard relations of a feedback amplifier.

Integrating eq. (11''), the open loop gain is given by

\begin{equation}
(15) \quad A = \exp(\pm i\alpha) \approx 1 + i\alpha,
\end{equation}

where $b = \alpha l$ ($l =$ thickness of the Kerr slab) and the Kerr medium has been taken as optically thin. Across the feedback path, by $z$-integration of the $q$-transform of eq. (11) ($\nabla^2_z \rightarrow -q^2$) we have a transfer function (taking unity reflectivity for all mirrors)

\begin{equation}
(16) \quad \beta = \exp \left[ -2\alpha \frac{q^2}{2k} \right].
\end{equation}

The field to be applied to the source term eq. (14) of the medium evolution is then given in terms of the external field $E_m$ as

\begin{equation}
(17) \quad |E|^2 = E_m^2 \left( 1 + 2\beta \alpha \frac{l}{L} \right).
\end{equation}

Introducing this into the $q$-transform of eq. (13), there results an average, $q = 0$,

\begin{equation}
(18) \quad \chi(q = 0) = \tau \chi_0.
\end{equation}

For $q \neq 0$, taking a time dependence as $\chi_t = \exp(\pm it)$, it is easily found

\begin{equation}
(19) \quad \chi = -\frac{1}{\tau} - Dq^2 + 2b\chi_0 \sin \frac{L}{2k} q^2
\end{equation}

which yields a threshold profile $I_t(q)$ of $I_m$ (take $\lambda = 0$) as in fig. 14a, in the scheme of fig. 13b, and bands of allowed oscillations (for $\lambda > 0$ at fixed input $I$) as in fig. 14b.

Similar results were originally derived by Firth[10] with the scheme of fig. 13b under the assumption that in the local intensity the interference term can be canceled, that is

\begin{equation}
(20) \quad I = |F + B|^2 = |F|^2 + |B|^2
\end{equation}

since the diffusion length $\sqrt{D\tau}$ is much larger than the optical wavelength $\lambda$.

43. Active optical morphogenesis. -- We consider a photorefractive crystal, pumped by an Argon laser beam and emitting within a ring cavity.

The geometry of the experiment is summarized in fig. 15. For a fixed-pump intensity and angle $\phi$, the dynamics is ruled by two control parameters, the d.c. field $E_{dc}$ applied across the crystal, in a direction parallel to the grating wave vector, and the aperture $a$ of the pupil of the cavity, which controls the Fresnel number $F$, that is the number $N = F^2$ of cavity modes. Here we call cavity modes the five-field configurations allowed by the boundary conditions.

Increase of the extensive parameter $F$, for fixed pump and $E_{dc}$, induces a series of bifurcations described in sect. 4. For low $F$ a single mode per time appears, but a given number $N$ of modes, limited by the relation $N \leq F^2$, competes sequentially. This means that each of the $N$ modes survives for a given time, and then it is replaced by another one and so on, in a periodic or chaotic alternation (PA and CA). Away from the narrow switching time intervals, the amount of mode mixing is negligible.

As $F$ increases further, pure cavity modes no longer alternate but they coexist at the same time with little mutual correlation, as shown by the shrinking of the correlation length. The many bright spots which are observed on the wave front or, equivalently, the intermingled dark spots giving rise to phase singularities, are no
crystal used provide a value $\tau = 1 s$, while values of $D$ reported in the literature are spread in the range $10^{-6} - 10^{-10}$ m$^2$/s.

We first isolate a fast varying part in $z$ and $t$ from the slow space-time dependence which is relevant for the dynamics, by writing

$$\vec{E} = E_p \exp(-i(k_0 z - \omega_0 t)) + E(r_1, z, t) \exp(-i(kz - \omega t)).$$

This way, we have extracted from the cavity field a plane wave along the cavity axis ($k$ along $z$), thus $E$ includes the full dependence on $r_1$ plus the residual slow dependence on $z$ and $t$.

The applied d.c. field in the $x$ direction induces a space charge drift, with a speed $v_d$, thus, from eq. (2), the scattering grating provides a frequency offset along $x$ given by

$$\Omega = v_d k_0 \theta.$$

Consequently, the cavity field has frequency $\omega$ detuned with respect to the pump frequency $\omega_0$ by

$$\omega = \omega_0 - \Omega.$$

We neglect this detuning for the following reason. In terms of the transverse wave number $q |q_{\text{max}}| \approx 100$ cm$^{-1}$, limited by the diffusive cut-off $q_{\text{cut}}^{-1} = 100$ $\mu$m of the crystal, propagation introduces a diffusive dephasing $\delta_1 = 3 q L / 2 k = O(1)$ since $L = 2.10^9$ cm.

Now, since the bandwidth of the active material is very narrow ($\Delta v \approx 1 / \tau \approx 18$ cm$^{-1}$) all oscillating modes are pulled within this bandwidth, so that the mutual dephasing between adjacent modes due to detuning is of the order of $\delta_2 = \Delta v L / c \approx 10^{-8}$ thus fully negligible in comparison to $\delta_1$.

In the Fourier expansion of the transverse dependence $r_1 \rightarrow k$, only the two components

$$\chi_0 \equiv \chi(k = 0)$$

and

$$\chi \equiv \chi(-K = k - k_0),$$

multiplied, respectively, by the cavity and pump fields, yield phase-matched contributions $P = \epsilon_0 \chi \vec{E}$. Thus the equation for the signal field reduces to

$$\left( \partial_t + \frac{i c}{2k} \nabla_1^2 \right) E = \frac{-i \omega}{2} \left( \chi_0 E + \chi \vec{E}_p \right) \text{rect}(z/l) = \left( -\frac{i \omega}{2} \chi \vec{E}_p - \vec{E} \right) \text{rect}(z/l),$$

Fig. 15. - Schematic set-up of active optics (pumped crystal emitting in a ring cavity).
where \( \text{rect}(z/l) = 1 \) only within the crystal thickness \( 0 < z < l \), and is zero elsewhere.

In writing the last line we have taken into account only the imaginary part of \( \chi_0 = \chi_0^i + \chi_0^h \) through \( \Gamma = -i(\omega/2)\chi_0^i \). This corresponds to a transmission loss of the crystal in the absence of pump which persists unmodified even with pump. The real part of \( \chi_0 \) induces a renormalization of \( \epsilon \) which is irrelevant for the field equation, since it has to be weighted by the cavity filling factor \( L/L \ll 1 \). Thus the material equation (21) provides the evolution for the envelope of the susceptibility

\[
\chi = \tilde{\chi} (\tau, t) \exp\{i(\kappa x - \Omega t)\}.
\]

By the usual argument of separation of fast and slow variables in the longitudinal dependence, matching of the phases which appear in the exponentials of eq. (24) and (22) accounts for the "fast" space variations on a scale

\[
2\pi / K = \lambda = \frac{\lambda}{\theta} = 30 \mu m \quad (\theta = 1/60, \lambda = 0.5 \mu m).
\]

We can rewrite the equation for the slow \( \chi \) envelope as

\[
(\partial_\tau + \frac{1}{\tau} - D\nabla^2)\tilde{\chi} = \frac{\chi E_0^* E}{E_p|E|^2} \left( 1 - \frac{|E|^2}{|E_p|^2} \right),
\]

where the source term is derived from the standard model of photorefractive media, with the assumption \( |E| \ll |E_p| \) and \( r \) is the complex photorefractive coefficient.

For the present purposes it is sufficient to keep only the linear part of this equation, thus we rewrite it as

\[
(\partial_\tau + \frac{1}{\tau} - D\nabla^2)\tilde{\chi} = \lambda E.
\]

The active system is somewhat different from the passive one. The main differences are the following:

i) within the BSO crystals, there are strong losses \( \Gamma = 1.9 \text{cm}^{-1} \), leading to a transmission of only 15% in one passage. Within the thickness \( l \) of BSO eq. (23) reduces to

\[
\partial_\tau E = -i\omega\tilde{\chi}E - \Gamma E \quad (\omega = \frac{1}{2} \frac{E_0}{E_p});
\]

ii) the diffusively thin approximation is not appropriate, since \( l = 1 \text{cm} \) (thinner crystals do not have enough gain to yield oscillations) against a diffusive length \( l_\epsilon = \sqrt{D\tau} \approx 100 \mu m \).

For the moment, let us consider a diffusively thin medium \( (l \ll \sqrt{D\tau}) \) so that \( \tilde{\chi} \) has no \( z \)-dependence. In such a case, eq. (26) can be easily integrated to provide

\[
E(1) = E(0) \exp\{-\Gamma t\} - \frac{i\kappa}{\Gamma} \tilde{\chi}(1 - \exp\{-\Gamma t\}).
\]

In the limit of an optically thick medium \( (\Gamma t > 1) \) this reduces to

\[
E(\tau, t) = -\frac{i\omega}{\Gamma} \tilde{\chi}(\tau, t).
\]

Thus, the output field carries no memory of the input field, but it shows the \( \chi \) dependence.

In the \( q \)-transform space, the free propagation of the output field across the length \( L > l \) is given by the solution of the \( t \)-independent \( q \)-transform of the source-free equation (23), which is

\[
\left( \partial_\tau + \frac{i}{2k} q^2 \right) \tilde{E}_q = 0,
\]

that is the field to be introduced into the material equation is

\[
\tilde{E}_q = \exp\left[-\frac{iL}{2k} q^2 \right] E_q(l) = \frac{-i\kappa}{\Gamma} \exp\left[-\frac{iL}{2k} q^2 \right] \tilde{\chi}.
\]

In the diffusively thin assumption, in the material equation (25) we have \( \nabla^2 \rightarrow \nabla_q^2 \rightarrow -q^2 \) thus we must solve the eigenvalue equation for \( \tilde{\chi}_q = \exp[iq\vec{q}_q] \) which is

\[
\lambda + \frac{1}{\tau} + Dq^2 = -\frac{i\kappa}{\Gamma} \exp\left[-\frac{iL}{2k} q^2 \right] \tilde{\chi}.
\]

**Fig. 16.** – Real part of the eigenvalue \( \lambda \) vs. \( q \) for active optics.
that is when we consider the real part of $\lambda$ we have a condition similar to eq. (19) and hence instability bands as in fig. 14b.

A more realistic approach, considering the correct optical and diffusive thickness including a $z$-integration, has been carried out[17] and it has provided a $\lambda$-$q$ diagram which looks like that of Turing morphogenesis, without allowed secondary bands (fig. 16).

5. Conclusion.

It is rather difficult to put an end to a review of work in progress. One area which has been heavily overlooked is that of morphogenesis in lasers. The main advantage of the systems here considered (Kerr and photorefractive media) is that their time scales are so slow that many transient features can be followed in real time and compared with the models, whereas in the laser case only time-integrated patterns are available thus far, and a breakthrough is necessary in order to reach a satisfactory match between models and experiment.

I have not mentioned possible technical applications as optical parallel computers, since in this regard there are still vague speculations rather than working schemes.

We have seen that the area of optical patterns is nowadays rapidly blossoming. Many aspects have already been explored, such as the onset of Turing patterns, the role of defects, the role of the boundary conditions, the scaling with the aspect-ratio (i.e., the Fresnel number of the confining cavity); however we are on the verge of producing results on these further items:

i) dynamics of pattern competition;

ii) role of hidden symmetries;

iii) localized structures, creation, translation, remotion;

iv) stabilization of a pattern as an extension of the strategies for the control of chaos.

REFERENCES