Quantum Morphogenesis: Spontaneous Transverse-Pattern Formation in the Wave Function of an Atomic Beam Interacting with a Light Field(*)

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Summary. — By coupling a microscopic quantum system to a dissipative field, a self-organized phase pattern emerges in the wave function of the microscopic system. The system is a supercooled atomic beam, crossing a laser beam, so that a local dephasing is induced on the wave function at each transverse coordinate, depending on the local field intensity. As the dephased wave function is propagated along a closed loop, diffraction provides a phase-to-amplitude conversion inducing local modifications of the atomic probability density. This in its turn modifies the field pattern. Thus, beyond a threshold controlled by the atomic intensity and the laser frequency, the uniform transverse phase of the wave function destabilizes towards a pattern. The symmetries of the pattern can be studied in terms of the symmetry changes induced on the laser beam. These spontaneous transverse patterns have nothing to do with the forced patterns induced by optical standing waves which define a preassigned confining potential, nor do they have to be confused with an alternative proposal for longitudinal-pattern formation (Zhang W. et al., Phys. Rev. Lett., 72 (1994) 60). The coupling here considered of a microscopic system obeying a Schrödinger equation for an atomic beam with a dissipative field obeying a diffusion equation appears as a generalization of the Hartree-Fock approximation.

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The phenomena of morphogenesis, or spontaneous pattern formation, initially illustrated via the competition of two chemical species (activator-inhibitor) in terms of coupled reaction-diffusion equations [1], have been extended to the interaction of

an optical field with a suitable medium in terms of a diffractive equation coupled to a
diffusive equation [2]. The diffractive equation, which approximates the wave
equation in the limit of a small Fresnel number (optical propagation confined mainly
in one direction), is formally equivalent to the Schrödinger equation which rules the
evolution of the wave function of an atomic beam. In this case, resonant exchange
between the energy of the atoms and an electromagnetic field induces changes in the
internal degrees of freedom, leaving unperturbed the translational degrees of
freedom. This amounts to a dephasing of the complex wave function. The formal
equivalence of the two equations suggests that many phenomena observed on the
Maxwell field have a counterpart for the atomic wave function, and thus one can
speak of «atom interferometry», «atom optics» and «atom diffraction» [3].

For instance, the dual version of the Young double-slit experiment [4] consists of a
supercooled atomic beam, with a de Broglie wavelength comparable with the optical
wavelength, crossing the standing wave of a laser beam detuned by \( \Delta = \omega_{12} - \omega_0 \) with
respect to the atomic transition frequency \( \omega_{12} \) so that a local dephasing \( \delta \psi(r) \) is
induced on the \( \psi \)-function at each transverse coordinate \( r \) depending on the local field
intensity \( |E(r)|^2 \):

\[
\delta \psi = (n(r) - 1) \frac{l}{v}.
\]

Here \( l \) is the size of the laser beam along the atomic path (interaction length), \( v \) is the
speed of the atomic beam,

\[
n(r) = 1 - \frac{m \omega_0}{\hbar^2 \kappa^2} U(r)
\]

is the pseudo refraction index for the propagating wave function, where the energy
shift \( U(r) \) is given in terms of the atomic-dipole moment \( \mu \) and the local intensity
\( n |f(r)|^2 \) (\( n = \) average photon number, \( f(r) = \) normalised field distribution) as

\[
U(r) = \frac{\omega_0}{\Delta} \frac{n}{2 \varepsilon_0 V_{\text{av}}} |f(r)|^2 \cdot \mu^2.
\]

Here, we start also from eqs. (1)-(3); however, the novel mechanism to be considered is the following (fig. 1):

i) The dephased \( \Psi \)-field is propagated along a closed loop (this is feasible, since
total-reflection atomic mirrors are nowadays available) and hence diffraction provides
a phase-to-amplitude conversion, whereby \( \psi(r) \) induces a local modification of \( |\Psi(r)|^2 \).

ii) The atomic probability density \( |\Psi|^2 \) modifies the stationary field pattern \( f(r) \)
(think, e.g., of an unstable equilibrium between two different field configurations
\( f_1(r) \) and \( f_2(r) \), disturbed by a very tiny atomic-population offset).

Thus, beyond a threshold controlled by the density of the atomic beam and by the
frequency position, the uniform transverse phase of the \( \Psi \) wavefront spontaneously
destabilizes towards a pattern. The symmetry of the pattern can be studied in terms of
the symmetry changes induced on the laser field.

In fact the energy shift \( U(r) \) appears as a spatially inhomogeneous optical
potential [4] to be introduced into the Schrödinger equation for the wave function
\( \psi(r, z, t) \).
Fig. 1. – Interaction between the wave function of an atomic beam and the laser standing wave propagating along \( z \) and confined within a narrow range \( l \) along \( z \). The forward-propagating wave function undergoes a dephasing as it crosses the laser region. The transmitted wave function is further dephased along the free propagation path, and it seems to the incident one in the laser volume, giving rise to a phase-to-amplitude conversion which modifies the laser intensity distribution, and hence the induced dephasing on \( \psi \). a) Forward and backward beams in a counterpropagating configuration; b) feedback configuration. The two configurations lead to the same result.

In the case of the forward beam (see fig. 2) we can write the wave function as

\[
\psi = \Psi(r, z, t) \exp \left( i(\omega t - kz) \right),
\]

where

\[
\omega_0 = \frac{\hbar^2 k^2}{2m}
\]

is the dispersion relation for a free particle. It is important not to confuse the plane-wave frequency \( \omega_0 \) corresponding to the translational degrees of freedom of the atom, with the frequency \( \omega_\text{int} \) corresponding to the internal degree of freedom (see fig. 2 and eq. (3)). The two-level optical transition has been treated separately using the formalism of ref. [4].

We consider the potential \( U \) localized over a \( z \)-interval so small that the main \( z \)-dependence in eq. (4) is included in the exponential factor.

The evolution equation for the wave function is

\[
\frac{i \hbar}{\partial t} \Psi = \frac{\hbar^2}{2m} \nabla^2 \Psi + U \Psi \text{rect} \frac{z}{l},
\]
Fig. 2. – Two-level atom (transition frequency $\omega_{12}$) interacting with a detuned laser field (optical frequency $\omega_0$). $r(x, y)$ are the transverse coordinates and $z$ the longitudinal one, for the atomic beam. $l$ is the longitudinal size of interaction region. The laser field is a standing wave made of two counterpropagating waves perpendicular to $z$ and with a pattern $f(r)$ in the atomic transverse directions $r$.

where $\text{rect}(z/l) = 1$ in the $z$-interval, of width $l$, where the laser is active and is identically 0 elsewhere.

Expanding $\nabla^2 = \nabla_z^2 + (\partial/\partial z)^2$ where $\nabla_z^2$ is the transverse Laplacian and taking for $\psi$ a slow $z$-dependence, so that

$$\frac{\partial^2 \psi}{\partial z^2} \ll k \frac{\partial \psi}{\partial z},$$

eq. (6) reduces to

$$\left( \frac{\partial}{\partial t} + y \frac{\partial}{\partial z} + i \frac{\hbar}{2m} \nabla_z^2 \right) \psi = -i \frac{\hbar}{l} U \psi \text{rect} \frac{z}{l}.$$

Equation (7) is formally equal to the equation describing transverse-pattern formation in an optical beam propagating along $z$ and with diffracting effects in the transverse direction $(x, y)$ [2].

By inspection, eq. (7) is a legitimate Schrödinger equation, without ad hoc approximations as, e.g., Hartree-Fock type, corresponding to consider $U$ as proportional to $|\psi|^2$.

However, the overall population density of the atomic beam, which is given by the superposition of incident and reflected wave functions (see fig. 1), that is

$$\rho(r, z) = |\psi_F + \psi_R|^2$$

can act as a source of perturbation for the laser field.

The laser system, being a macroscopic dissipative system, is appropriately described by a reaction-diffusion equation. Since $U$ is proportional to the local laser
intensity, we can write its equation as

\begin{equation}
\left(\tau_c \frac{\partial}{\partial t} + 1 - \frac{i}{\hbar} \nabla^2\right) U = \alpha + \beta \rho. \tag{9}
\end{equation}

Here \( \alpha \) and \( \beta \) are suitable constants, and \( \rho \) is the local density of the atomic beam crossing the hatched space region in fig. 1, where the laser field is confined. Equation (9) does not account for all the detailed processes which occur in a laser, it is just a macroscopic equation describing how the laser intensity \( U \) is affected by the atomic density.

The unperturbed stationary laser intensity in the absence of the atomic beam is \( \alpha = U_0 \).

The coupling of eqs. (7) and (9) gives rise to a self-organized pattern formation. In order to study the onset of a non-uniform \( x \)-component of \( \Psi \), we proceed by linear stability analysis.

Outside the interaction region, eq. (7) reduces to

\begin{equation}
\left( \frac{\partial}{\partial z} - i \frac{\hbar}{2mu} \nabla^2 \right) \Psi = 0 \tag{10}
\end{equation}

and inside the laser medium, neglecting the diffraction effects over the length \( l \), it can be approximated as

\begin{equation}
\frac{\partial}{\partial z} \Psi = - \frac{i}{\hbar} U \Psi, \tag{11}
\end{equation}

which yields

\begin{equation}
\Psi = \exp \left[ -i \frac{\pi U}{\hbar} \right] \Psi_{in} \approx \left( 1 - i \frac{\pi U}{\hbar} \right) \Psi_{in}, \tag{12}
\end{equation}

where \( \tau = l/\nu \) is the atomic transit time across the laser. Propagating the beam over a length \( L \), eq. (10) yields a phase factor for the \( q \) Fourier component (\( q \) being the Fourier transform of the transverse coordinate \( z \)) given by [5]

\begin{equation}
\Psi(q, z = L) = \exp \left[ -i \frac{\hbar}{2m} T q^2 \right] \Psi(q, z = 0), \tag{13}
\end{equation}

where \( T = L/\nu \) is the round-trip time through free space.

Combining eqs. (12) and (13) in a feedback configuration, as shown in fig. 1, we have a phase-to-amplitude conversion. Precisely, the atomic density interacting with the laser medium is given in terms of the impinging density \( \rho_{in} = \left| \Psi_{in} \right|^2 \), by

\begin{equation}
\left| \Psi \right|^2 = \rho_{in} \left( 1 + \frac{2\tau}{\hbar} U \sin \frac{\hbar T}{2m} q^2 \right). \tag{14}
\end{equation}

Indeed, in a feedback approximation with a forward transmission \( A \) given by eq. (12) and a backward transmission \( B \) given by eq. (13), the renormalized input wave
function is given by

\[ \mathcal{V}_\infty \frac{1}{1 - BA} = \mathcal{V}_\infty (1 + BA) \mathcal{V}_\infty (1/(1 - BA)) = \mathcal{V}_\infty (1 + BA) \]

and its modulus squared yields eq. (14) for each \( q \)-component. Introducing eq. (14) into eq. (9) we have a \( q \neq 0 \) perturbation of the \( U \)-field ruled by, for a time dependence \( U_q = \exp(\lambda t) \), the \( \lambda - q \) relation

\[ \tau \lambda + 1 + l_0^2 q^2 = \beta \varphi_{\infty} \frac{2 \pi}{\hbar} \sin \frac{hT}{2m} q^2. \]

For \( \lambda = 0 \), we have the corresponding threshold \( \rho_{\infty}^l \)

\[ \rho_{\infty}^l = \frac{1 + l_0^2 q_0^2}{\beta \frac{2 \pi}{\hbar} \sin \frac{hT}{2m} q^2}. \]

This is minimum for

\[ \frac{hT}{2m} q_0^2 = \frac{\pi}{2} + l \cdot 2 \pi \]  \[ (l \text{ integer}). \]

The value of \( q_0 \) is given in the appendix. A \( q \neq 0 \) pattern in \( U \) corresponds to a change of stability on the laser standing waves (e.g., the transition from one to another longitudinal laser mode, i.e. along \( x \)) resetting the modes of the field pattern, and hence, through eq. (12), the \( \mathcal{V} \) phase. Figure 3 gives a qualitative plot of threshold density and eigenvalue \( \lambda \) vs. \( q^2 \).

The minimum density threshold for no diffusion in the laser medium \( (l_0 = 0) \) is given by

\[ \rho_{\infty}^l = \frac{\hbar}{2 \beta \tau}. \]

The \( \beta \)-factor has to be guessed by some straightforward calculations given in the appendix. It results that a perturbation of \( 10^5 \text{ atoms/cm}^3 \) is sufficient to destabilize the uniform wavefront of the \( \mathcal{V} \)-function and induce a pattern.

As a conclusion, spontaneous pattern formation in the phase of a \( \mathcal{V} \)-function is easily achievable with presently available laboratory techniques.

This spontaneous pattern has not to be confused with the forced pattern created on purpose by confining the atoms in suitably engineered potential wells [6].

An alternative proposal for longitudinal-pattern formation has been recently formulated [7]. Our calculations show that our Turing transverse patterns are feasible, and they should be easily detectable.

Perturbation of an ideal plane wave, as done in Turing pattern calculations, corresponds to what realized in optics with coherent laser impinging onto a non-linear dephasing material, as shown in ref. [2]. In the atomic-beam case, careful consideration must be taken of the coherence limits, as shown in appendix A2. While transverse coherence does not impose strict limitations, longitudinal coherence
Fig. 3. – Threshold atomic density (a)) and eigenvalue (b) vs. $q$ for atomic morphogenesis.

--- $\rho^1 = 20 \text{ a.u., } \lambda^1 = 50 \text{ a.u.}$ In order to fix the longitudinal scale, $q$ has been rescaled to $q_0$, as given by eq. (17).

imposes a free propagation length of the order of the de Broglie length. One should indeed consider that the quadratic dispersion relation eq. (5) gives a $\Delta \omega$ increasing with the average velocity of the atoms and thus no advantage is obtained if one superposes a global velocity to the thermal spread (see eq. (A.15)).

A re-scrambling of transverse atomic density implies a replacement of atoms, and thus it is limited by an uncertainty relation $\Delta x \Delta p_x \geq \hbar$, where $\Delta p_x = \hbar \Delta q$ is the transverse-momentum transfer.

Finally, it is worth noticing that in the limit of a very fast and highly diffusing medium, eq. (9) provides a local relation $U = \beta \rho$. Replacing this into the Schrödinger equation, we have the standard Hartree-Fock approximation.

APPENDIX

A’1. Conversion factors and threshold density. – Let us consider alkali atoms with an allowed dipole transition of matrix element $\mu = e r = 10^{-22} \text{ C cm}$. Taking an optical frequency $\omega = 10^{16} \text{s}^{-1}$, a detuning $\Delta = 10^9 \text{s}^{-1}$ and a laser volume $V = 1 \text{ cm}^{-2}$, the
conversion factor from $U$ to $n$ is (cfr. eq. (3) with $|f|^2 = O(1)$)

$$U \approx 10^{-30} \, n,$$

where $U$ is given in joules.

The threshold density for the onset of an instability in $U$ is given by eq. (16)

$$\rho^t = \frac{\hbar}{2\beta \tau^t}.$$

Here the transit time $\tau = l/v$ is easily evaluated. For atoms of $M = 50$ a.m.u. cooled at $T = 100$ $\mu$K, the thermal velocity is $v \approx 10$ cm/s so that $\tau \approx 0.1$ s.

But $\beta$ is unknown. On the other hand, we can write simple-minded rate equations for the joint evolution of the photon number $n$ and the atomic population $N$ within the laser cavity, that is

$$\begin{align*}
\frac{dn}{dt} + \frac{n}{\tau_c} &= BnN, \\
\frac{dN}{dt} &= -\gamma(N - N_0) - BnN,
\end{align*}$$

where for alkali atoms, the Einstein coefficient has a value $[8] R = 1$ s$^{-1}$, the cavity decay time is $\tau_c = 10^{-7}$ s (30 transits in a 100 cm long cavity), the atomic-decay rate is $\gamma \approx 10^9$ s$^{-1}$, and $N_0$ is the pump value. If we introduce the threshold value for population

$$\overline{N} = \frac{1}{B\tau_c} \approx 10^7$$

and the saturation number for photons

$$n_s = \frac{\gamma}{B} \approx 10^8,$$

then the steady photon number is given by

$$\bar{n} = n_s \left( \frac{N_0}{\overline{N}} - 1 \right) = 10^{-3} n_s = 10^6,$$

having taken a pump/threshold ratio equal to $N_0/\overline{N} \approx 1.01$.

The density of inverted threshold ratio equal to $N_0/\overline{N} \approx 1.01$.

The density of inverted atoms which starts the laser process for a cavity value $V = 1$ cm$^3$, is given by

$$\bar{\rho} = \frac{\overline{N}}{V} \approx 10^7 (\text{cm}^{-3}).$$

We now consider the perturbation

$$\begin{align*}
n = \bar{n} + \tilde{n}, \\
N = \overline{N} + \tilde{N}.
\end{align*}$$
From eq. (A.3) the perturbed photon equation is given by

\[ \frac{d}{dt} + \frac{1}{\tau_c} \tilde{n} = B(\tilde{N}\tilde{n} + \tilde{n}\tilde{N}) \approx 10^7 \tilde{n} + 10^6 \tilde{N} \]

or \( \tau_c \frac{d}{dt} \tilde{n} \approx 0.1 \tilde{N} \).

We compare this with the perturbed equation (9) for \( \tilde{U} = U - U_0 \) in the case of no diffusion \( (u_0 = 0) \)

\[ \tau_c \frac{d}{dt} \tilde{U} = \beta \tilde{\varphi} \]

Accounting for the conversion factor of eq. (A.1), and with \( \tilde{\varphi} = \tilde{N} \) in \( \text{cm}^3 \), (since \( V = 1 \text{ cm}^3 \)), it results

\[ \beta = 10^{-35} \text{ (J cm}^3\text{).} \]

Replacing this in eq. (A.2), we have

\[ \rho^4 = \frac{10^{-34}}{2 \cdot 10^{-35} \cdot 0.1} \left( \frac{J}{\text{s} \text{cm}^3} \right) = 10^2 \text{(cm}^{-3}\text{).} \]

A2. Transverse and longitudinal atomic coherence. – Several experimental configurations can be adopted. E.g., one can prepare a bunch of atoms (optical molasses) at a low temperature \( T = 1 \mu \text{K} \) and then push the atoms upward by another laser, cross the dephasing beam and return by gravity (in this case the reflection does not need a mirror), or use a supersonic beam, with an average velocity \( v \) much longer than the velocity spread \( \tilde{v} \) of the beam.

For simplicity, we think of an experiment in the absence of gravity, with the atoms selected out of a cooled molasses by a diaphragm of diameter \( d = 0.1 \text{ cm} \), in a geometry as in fig. 1.

For Cs atoms \( (m = 133 \text{ a.m.u.} \approx 3 \cdot 10^{-25} \text{kg}) \) at \( T = 1 \mu \text{K} \), the thermal speed is

\[ \tilde{v} = \sqrt{\frac{kT}{m}} = 0.5 \cdot 10^{-2} \text{ ms}^{-1} \]

giving the thermal de Broglie length

\[ \lambda_T = \frac{2 \pi \hbar}{m \tilde{v}} = 4 \cdot 10^{-7} \text{ m.} \]

a) Transverse coherence.

If we consider the atomic emission as a black-body radiator with an aperture \( d = 0.1 \text{ cm} \), then the transverse-coherence relation in order to «fill» coherently the interferometer made of a laser standing wave, with consecutive nodes separated by \( \lambda/2 \) \( (\lambda = \text{optical wavelength}) \) placed at a distance \( R \), is given by [9]

\[ \frac{\lambda_T}{d} > \frac{\lambda}{2R}. \]

The relation says that the diffraction angle corresponding to a «single mode» of the
wave function has to fill the geometric angle $\lambda/2R$. It provides

$$R > d \frac{\lambda}{2\lambda_T} \equiv 0.1 \text{ cm}$$

which is easily fulfilled in the experiment.

b) Longitudinal coherence.

The backward wave function, after a transit time $T = L/v$ has to be in phase with the forward one. Thus, $T < T_{coh}$, where

(A.12) $$T_{coh} = \frac{1}{\Delta \omega}.$$ 

The linewidth associated with the thermal spread is

(A.13) $$\Delta \omega = \frac{\tilde{v}}{V},$$

where, from eq. (5),

(A.14) $$\omega = \frac{1}{\hbar} \frac{m \tilde{v}^2}{2}.$$ 

Thus we have eventually a condition for the free propagation length $L$ which does not imply the global speed $v$, but just the thermal fluctuation $\tilde{v}$,

(A.15) $$L < v T_{coh} = v - \frac{2h}{m \tilde{v}^2} - \frac{2h}{m \tilde{v}^2} = \frac{2 \cdot 10^{-8}}{3 \cdot 10^{-23} \cdot 0.5 \cdot 10^{-2}} \approx 1.5 \cdot 10^{-7} \text{ m}.$$ 

Equation (A.15) puts a too stringent requirement. In fact, the coherence length is of the order of the optical wavelength. We then must cool further or reduce the mass. We could gain a factor 32 going to helium atoms, and thus the atomic beam can cross a laser beam waist with $l \sim L/2 \sim 16$ optical wavelengths.

Anyway, a longitudinal-coherence length $L$ of the order of the de Broglie length as given by eq. (A.15), once replaced in eq. (17) for $T = L/v$, yields a critical $q_0 = \sqrt{2m/kT} = 2\pi/\lambda_T$.

REFERENCES