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Adaptive recognition and control of chaos

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Abstract

Exploiting the information provided by the local variation rates of a dynamical system, we show how an adaptive algorithm is able to recognize and stabilize the unstable periodic orbits embedded in a chaotic attractor, through additive corrections to the dynamics. The relative weight of each correction is adjusted upon the size of the local expansion rate. Application of this method to chaotic Lorenz and four dimensional Roessler systems is discussed.

There are two different methods for inspecting a (possibly continuous) dynamical system: the standard one whereby one selects a regular series of observation times (i.e., separated by a constant time interval) and plots the evolution of the corresponding positions measured in some coordinate space. When considering such a geometrical series of data, one decides whether a signal is regular or chaotic or random.

An alternative approach [1] consists in fixing a narrow observation window in the coordinate space and extracting those extrema of the signal lying within the window. In this case, the arising geometric positions are a clustered set, but the sequence of return times to the window is erratic when the considered dynamics is irregular. This latter recognition method ceases to be useful in the case of multibranched attractors, as e.g. the Lorenz case, which include sudden jumps from one branch to the other. Indeed, if the window selects positions of one branch, time sequences are affected by wide gaps in correspondence to the flow jumps; thus the stroboscopic series provides only partial information on the dynamics.

In this report we will summarize the results of recent studies [2,3], which have introduced an adaptive strategy for recognition and control of chaos in dynamical systems. Namely, we will show how to implement a stroboscopic inspection which overcomes the above difficulty, through an adaptive windowing controlled by the local variation (expansion or contraction) rates. This not only provides useful indicators for chaos recognition, but it can also be applied for chaos control even for systems displaying more than one positive Liapunov exponent.

Controlling chaos consists in perturbing the considered systems in order to stabilize a given unstable periodic orbit (UPO) embedded in the chaotic attractor [4].

The first control method proposed by Ott, Grebogi and Yorke (OGY) [5] consists in slight readjustment of a control parameter each time the trajectory crosses the Poincaré section (PS). Since a generic UPO is mapped on the PS by an ordered sequence of crossing points, OGY is able to stabilize such a sequence whenever the

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chaotic trajectory visits closely a neighborhood of one of the desired UPOs points. It can occur that the time lapse for a natural passage of the flow within the fixed neighborhood (hence for switching on the control process) is very large. To minimize such a waiting time, a technique of targeting has been also introduced [6].

Another technique to constrain a nonlinear system $\boldsymbol{x}(t)$ to follow a prescribed goal dynamics $\boldsymbol{g}(t)$ has been introduced [7] based upon the addition to the equation of motion $\mathrm{d}\boldsymbol{x}/\mathrm{d}t = \boldsymbol{F}(\boldsymbol{x})$ of a term $\boldsymbol{U}(t)$ chosen in such a way that $|\boldsymbol{x}(t) - \boldsymbol{g}(t)| \to 0$ as $t \to \infty$. Ref. [7] considers $\boldsymbol{U}(t) = \mathrm{d}\boldsymbol{g}/\mathrm{d}t - \boldsymbol{F}(\boldsymbol{g}(t))$. However, even though the method provides robust solutions, in general the perturbation $\boldsymbol{U}$ to be done is of the same order of magnitude as the unperturbed dynamics $\boldsymbol{F}$.

In other papers the effects of periodic [8,9] and stochastic [10] perturbations are explored in producing dramatic changes in the dynamics, which however are quite difficult to predict and in general are not goal oriented.

A further method [11] has been proposed, based on a continuous application of a delayed feedback term in order to force the dynamical evolution of the system toward the desired periodic dynamics whenever the system visits closely such a periodic behavior.

On the other hand, many experimental systems have been studied with the aim of establishing control over chaos.

Experimental chaos control and higher order periodic orbit stabilization have been successfully done in dealing with a thermal convection loop [12], a yttrium iron garnet oscillator [9], a diode resonator [1], an optical multimode chaotic solid-state laser [13], a Belouzov–Zabotinsky reaction diffusion chemical system [14], a CO_2 single mode laser with modulation of losses [15]. In most cases stabilization of UPOs was achieved by the technique of occasional proportional feedback (OPF) introduced in Ref. [1].

Nevertheless, the possibility of controlling high dimensional systems with more than one positive Liapunov exponent (hyperchaos) has not yet been fully explored. Indeed, even though a discrete hyperchaotic dynamics (i.e. ruled by a map) has been controlled [16] and targetted [17], extension of the above methods to continuous dynamics is still an opened question.

Here we present a new method which intervenes on a adaptive time scale T_s intermediate between the short resolution time T_1 of continuous methods [11] and the long one T_2 corresponding to the delay between two successive PS crossing used in Ref. [5].

Let us consider a general dynamical dissipative system ruled by

$$\dot{\boldsymbol{x}} = \boldsymbol{G}(\boldsymbol{x}, \mu), \tag{1}$$

where $\boldsymbol{x}$ is a D-dimensional vector, $\boldsymbol{G}$ a nonlinear function and μ a set of control parameters chosen in such a way as to produce chaos.

For each component $i (i = 1, 2, \ldots, D)$ of $\boldsymbol{x}$ and at the time $t_{n+1} = t_n + \tau_n$ (τ_n being the adaptive observation interval to be later specified), one defines the variation over the nth observation time interval τ_n as

$$\delta x_i(t_{n+1}) = x_i(t_{n+1}) - x_i(t_n), \tag{2}$$

and the local variation rates λ's of δx_i as

$$\lambda_i(t_{n+1}) = \frac{1}{\tau_n} \log \left| \frac{\delta x_i(t_{n+1})}{\delta x_i(t_n)} \right|. \tag{3}$$

By use of λ's, it is possible to fix the new observation at the time $t_{n+2} = t_{n+1} + \tau_{n+1}$, where

$$\tau_{n+1} = \min_i \; \tau_{n+1}^{(i)}$$

and

$$\tau_{n+1}^{(i)} = \tau_n^{(i)}(1 - \tanh(g\lambda_i(t_{n+1}))). \tag{4}$$

Eq. (4) arises from the following considerations. In order to obtain a series of δx_i ranging over a small interval, we contract (expand) the observation time interval whenever the actual value of δx_i is bigger (smaller) than the previously observed one. This way, the observer performs a stroboscopic task, whereby the resulting geometric sequence of δx_i is clustered around a point, while the sequence of observing times is now irregular whenever the motion is chaotic.

The method exploits the information provided by the local variation rates, shrinking or stretching the next observation interval in order to minimize the variation in width of the window containing the coordinates of the two end points. This depends crucially on the fact that expansion or contraction in a given direction i cannot be monotonic in course of time for an attractor confined within a finite support, where the trajectory undergoes frequent twistings.

The hyperbolic tangent function maps the whole range of $g\lambda_i$ into the interval $(-1, +1)$, while the constant g, strictly positive, has to be chosen in such a way as to forbid $\tau_n^{(i)}$ from going to zero. It may be taken as an a priori sensitivity. However, a more sensible assignment is discussed in Ref. [2], which consists in looking at the unbiased dynamical evolution for a while and then taking a g value smaller than the reciprocal of the maximal λ recorded in that time span. The two strategies have counterparts in neural networks [18]. Indeed, while choosing a fixed g is like adjusting the connectivities of a neural network by a preliminary learning session, adjusting g upon the information accumulated over a given number of previous observations corresponds to considering g as a kind of long-term-memory, as opposed to the short-term-memory represented by the sequence of τ_n.

Also the strategy of adjusting to the fastest variation by the selection of the minimum $\tau_{n+1}^{(i)}$'s at each time seems to be appropriate in many physiological operations done by living organisms [18], and it has been implemented in a neural network approach with applications to real time Monte–Carlo calculations as well as to real time recognition of particle tracks in high energy experiments [19].

The sequence of stroboscopic times (obtained starting from t_0, $\tilde{\tau}$ and $\delta\bar{x}(t_0)$) $t_0, t_1 = t_0 + \tilde{\tau}, t_2 = t_1 + \tau_1, \dots, t_{n+1} = t_n + \tau_n, \dots$ now contains the relevant information on the dynamics. Thus, characterization and recognition of chaos can be done by the study of such a time sequence.

In the following we will summarize the application of such a method to the chaotic Lorenz (Lo) [20] model and to the 4-dimensional Roessler (Ro4) [21] model. The latter one consists in a 4-dimensional dynamical system described by the equations:

$$\begin{aligned} \dot{x}_1 &= -x_2 - x_3, \\ \dot{x}_2 &= x_1 + 0.25x_2 + x_4, \\ \dot{x}_3 &= 3 + x_1x_3, \\ \dot{x}_4 &= -0.5x_3 + 0.05x_4. \end{aligned} \tag{5}$$

For initial conditions $x_1(0) = -20, x_2(0) = x_3(0) = 0, x_4(0) = 15$, and for the values of control parameters displayed in the equations, the system (5) produces a hyperchaotic dynamics with two positive Liapunov exponents [21].

Looking at Eq. (4), when $|g\lambda_i| \ll 1$, two successive τ_n must be strongly correlated, even though the τ distribution may be spread over a rather wide support. As a consequence, the map τ_{n+1} vs. τ_n must cluster along the diagonal and any appreciable deviation from the diagonal denotes the present of uncorrelated noise. In Fig. 1 we plot the return map τ_{n+1} vs. τ_n for Ro4 and Ro4 with 1% of additional noise.

Further analysis on the τ sequence and on λ sequence has been carried out in Ref. [2] where the β indicator has been introduced as a suitable discriminator between determinism and stochasticity.

Here we are interested in stabilizing a periodic dynamics, so that we need to extract the periods of UPOs embedded in the chaotic attractor. For this purpose, instead of considering the single step map, we construct the maps τ_{n+k} vs. $\tau_n, k = 1, 2, \dots$ and we plot the RMS η of the point distribution around the diagonal of such maps as functions of the step interval k.

Since for a chaotic dynamics temporal self-correlation lasts only for a finite time, one should expect to obtain a monotonically increasing function. In fact the chaotic dynamics brings the trajectory in the phase-space to shadow neighborhoods of different UPOs. As the trajectory gets close to an UPO of period T_j, temporal self-correlation is rebuilt after T_j and the distribution of τ includes windows of correlated values appearing as minima of η vs. k around $k_j \cong T_j/\langle\tau\rangle$, $\langle\tau\rangle$ being the average of the τ distribution.

To give an example, we report in Fig. 2 the $\eta - k$ plot for Ro4, from which one can extract the different UPOs periods looking to the minima of the η curve. Indeed, since during the stroboscopic observation the time intervals are changing, a rigorous determination of the period is provided by looking at the cost function in the vicinity of the minima of the η curve.

Let us call $\tau_{\min}$ and $\tau_{\max}$, respectively, the minimum and the maximum τ values during the recognition task. It is evident that the period T_j of the jth UPO is such that

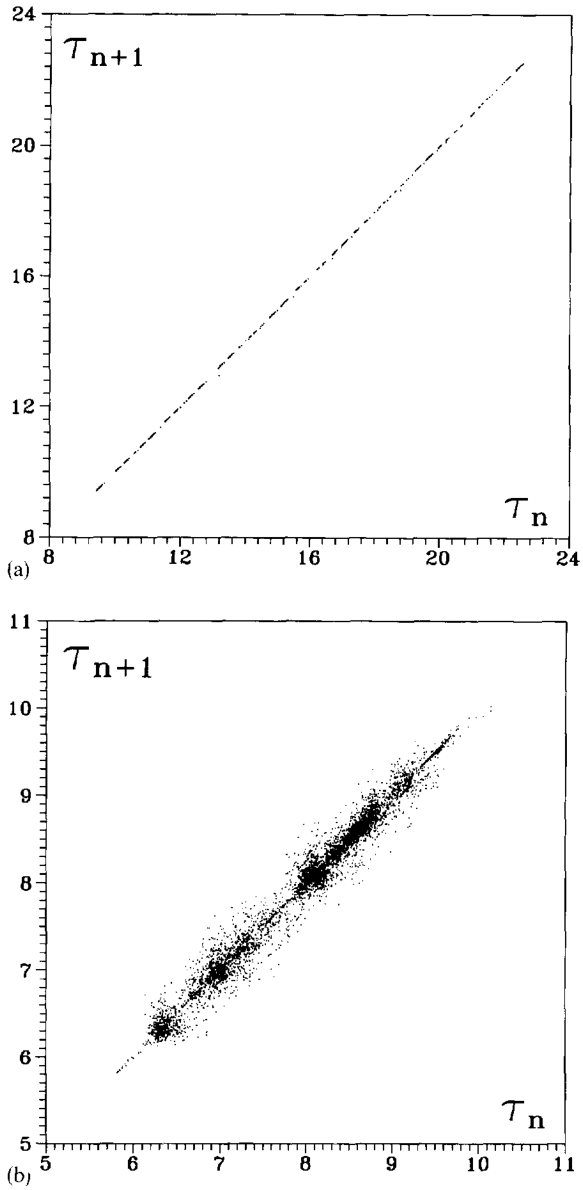

Fig. 1. Return maps τ_{n+1} vs. τ_n for (a) Ro4 and (b) Ro4 with an additional 1% white noise. Initial conditions: $x_1(0) = -20, x_2(0) = x_3(0) = 0, x_4(0) = 15$ as discussed in the text. $g = 0.000048$. Vertical and horizontal axes have to be multiplied for 10^{-3}.

$$k_j \tau_{\min} \le T_j \le k_j \tau_{\max},$$

where k_j is the jth minimum of the η curve. Thus, introducing a cost function

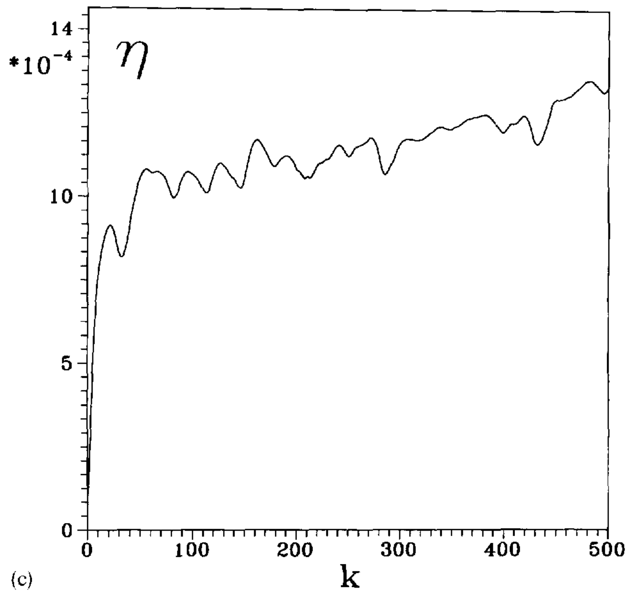

Fig. 2. The η–k plot for Ro4 attractor. Initial conditions and control parameters as in the caption of Fig. 1. The recognition task has been performed with $g = 0.01$. Vertical axis has to be multiplied for 10^{-4}.

$$C(\nu) = \sum_{n=1}^{N} |\boldsymbol{x}(t_n) - \boldsymbol{x}(t_n - \nu)|, \tag{6}$$

where the sum runs over the N data recorded during the recognition task, and looking for the minimum of $C(\nu)$ with ν running from $k_j \tau_{\min}$ to $k_j \tau_{\max}$, one obtains the real period T_j of the jth UPO of the dynamics under study.

Fig. 3 shows the cost function $C(\nu)$ for the eighth minimum of the η plot. $\nu = 54.64$ measures the value of the period-8 UPO of Ro4.

When periods $T_j (j = 1, 2, \ldots)$ of the UPOs have been measured, stabilization of each one can be achieved when the system naturally visits closely phase space neighborhoods of that UPO.

For a nonautonomous system it can occur that a period T corresponds to many degenerated UPOs. In such a case, selection of the desired one can be provided by the study of the topology of all UPOs corresponding to the same period and by switching on the control task when the system is shadowing the chosen UPO to be stabilized. Some topological approaches to the UPOs detection are contained in Ref. [22].

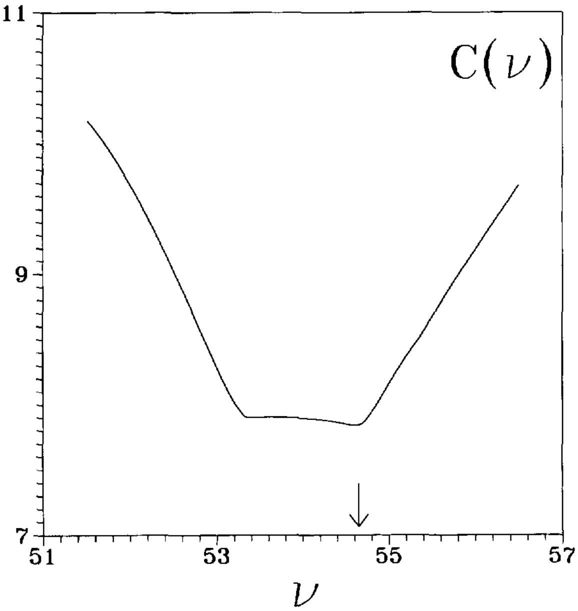

Fig. 3. Cost function $C(\nu)$ for the eighth k minimum of Fig. 2. $\nu = 54.64$ (indicated by the arrow) is the measurement of the period for one of the orbit 8 of Ro4.

The control procedure is done by use of the following modified algorithm. At each new observation time $t_{n+1} = t_n + \tau_n$ and for each component i of the dynamics, instead of Eq. (2), we evaluate the differences δ between actual and desired values:

$$\delta x_i(t_n + 1) = x_i(t_{n+1}) - x_i(t_{n+1} - T_j), \tag{7}$$

and the local variation rates λ's now read

$$\lambda_i(t_{n+1}) = \frac{1}{\tau_n} \log \left| \frac{x_i(t_{n+1}) - x_i(t_{n+1} - T_j)}{x_i(t_n) - x_i(t_n - T_j)} \right|. \tag{8}$$

Eq. (4) and choice of the minimum are kept for the updating process of τ's.

Defining $\boldsymbol{U}(t)$ as the vector with ith component (constant over each observation time interval) given by

$$U_i(t_{n+1}) = \frac{1}{\tau_{n+1}}(x_i(t_{n+1} - T_j) - x_i(t_{n+1})), \tag{9}$$

we add such a vector to the evolution equation, which now reads

$$\frac{\mathrm{d}\boldsymbol{x}}{\mathrm{d}t} = \boldsymbol{G}(x, \mu) + \boldsymbol{U}(t). \tag{10}$$

Now, λ's are measuring how the separation of actual trajectory from desired one evolves; indeed, λ negative means that locally the true orbit is collapsing into the desired one and hence the actual dynamics is shadowing the desired UPO. On the contrary, λ positive implies that the actual trajectory is locally diverging away from the desired one and control has to be performed in order to constrain the orbit to shadow the desired UPO.

As a consequence, contraction or expansion of τ's now reflects the necessity to perturb the dynamics more or less often in order to stabilize the desired UPO, as well as it fixes the weight of the correction to be done, which, once a given T_j has been chosen by the operator, is selected by the same adaptive dynamics. Indeed, integrating Eq. (10) from t_{n+1} to t_{n+2}, since $\boldsymbol{U}(t)$ is constant over τ_{n+1}, it yields the term $\boldsymbol{x}(t_{n+1} - T_j) - \boldsymbol{x}(t_{n+1})$ which corrects for the previously observed chaotic deviation from the goal dynamics.

Once again, the introduced adaptive weighting procedure in Eq. (9) assures the effectiveness of the method (perturbation is larger or smaller whenever it has to be) as well as the fact that the additive term $\boldsymbol{U}$ is much smaller than the unperturbed dynamics $\boldsymbol{G}$.

In Fig. 4 we show the control of period-8 of Ro4 and of period-5 of Lo. In the last case a bigger initial δ has been selected in order to highlight the shadowing process.

Finally, in Fig. 5 we report the perturbation $U_1(t)$ and the unperturbed dynamics G_1 for the Ro4 model during the control task of period-8, in order to show that the former is between two and three orders of magnitude smaller than the latter as expected by the above discussion.

As for time scales, notice that while in Eq. (7) differences δx_i are evaluated with respect to the goal dynamics (thus over the period T_j), Eq. (8) all λ's are evaluated over the adaptive τ, which, as discussed in Ref. [2] has to be much larger than the Runge–Kutta integration interval (about 100 times larger) but much smaller than the UPOs period (as it is evident from Fig. 2 where all k_j's are much above the unity). This way the method introduces a natural adaptation time scale intermediate between minimum resolution time and time scale of the periodic orbits.

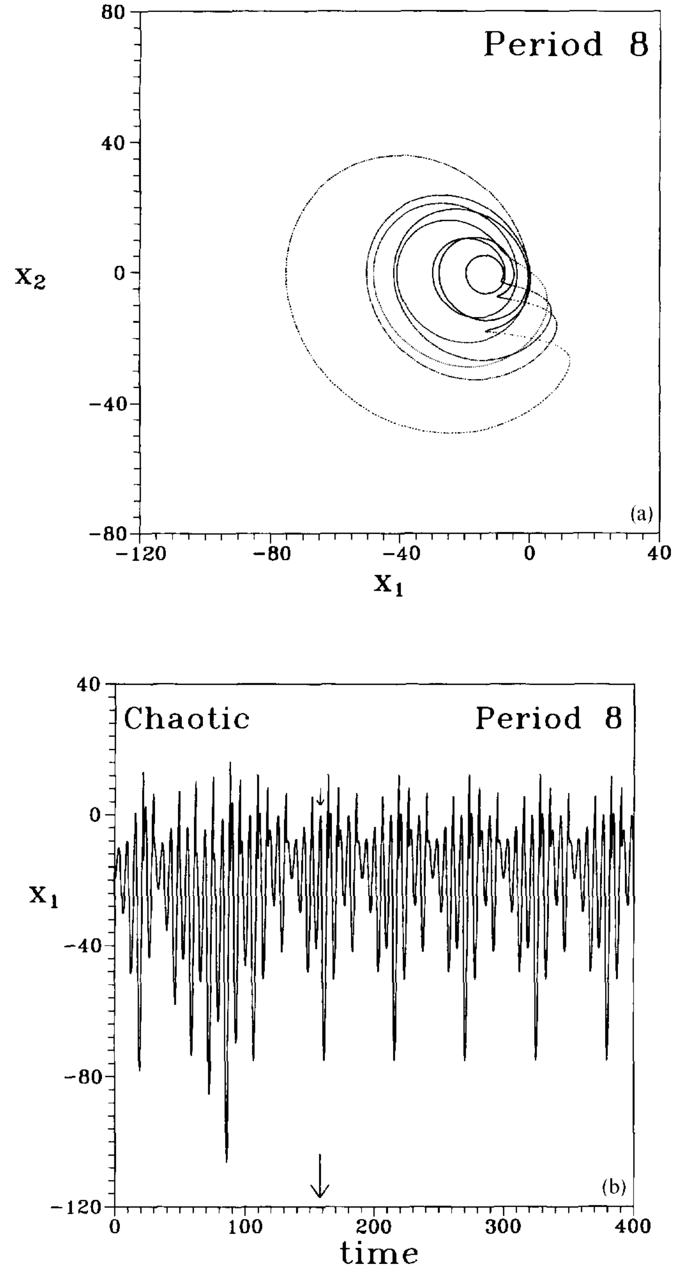

Fig. 4. (a) (x_1, x_2) projection of the phase space portrait for the controlled period-8 of Ro4 attractor. Control task has been performed with period-8 extracted from Fig. 3 and $g = 10^{-5}$. (b) Time evolution of the first component x_1 of Ro4 before and after control. Arrows indicate the instance at which control task begins. (c) $(x, \dot{x})$ representation for the period-5 of Lo ($\sigma = 10, b = 8/3, r = 25$). In this latter case control task has been provided with period-5 measured by the minimum of the cost function ($T = 6.09$), $g = 10^{-5}$, but the control has been switched on when the distance between true orbit and desired period was quite large in order to highlight the shadowing process.

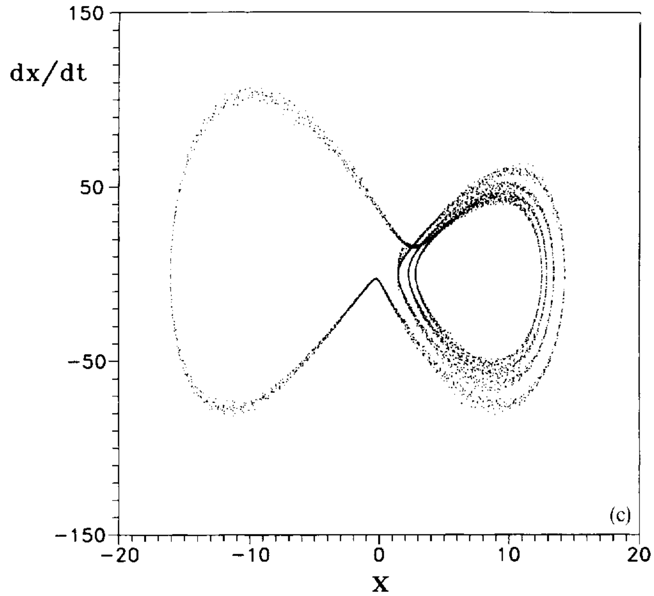

Fig. 4. (continued)

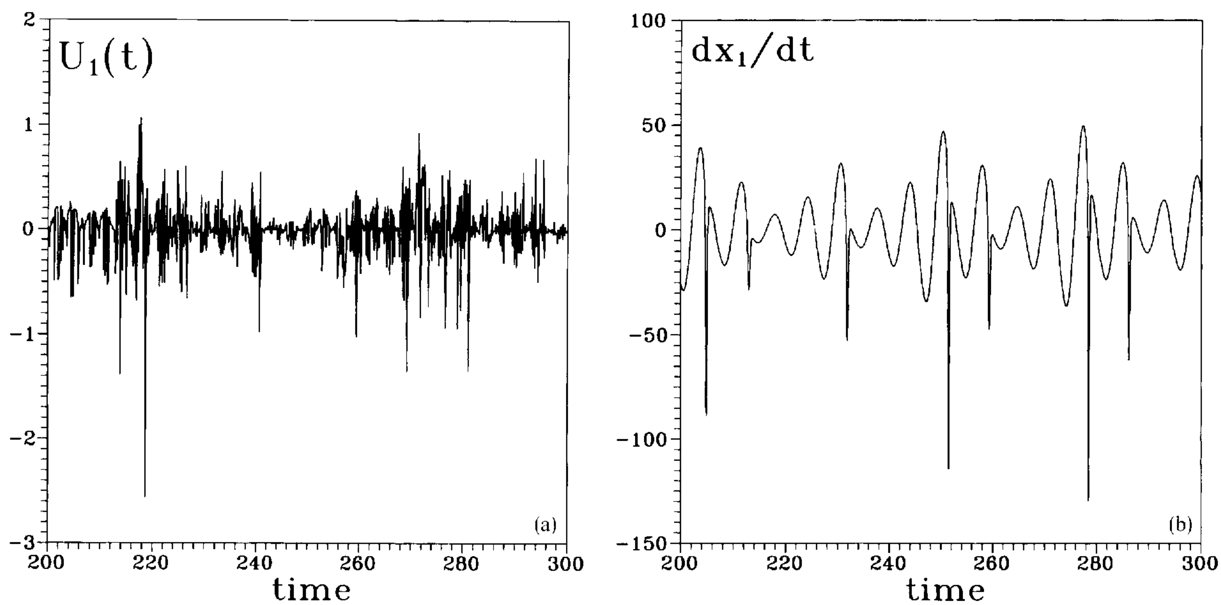

Fig. 5. (a) Temporal evolution of the first component of the additive controlling term U_1 during the control of period-8 Ro4 and (b) temporal evolution of the uncontrolled dx_1/dt. The adaptive correction term is between two and three orders of magnitude smaller than the natural evolution of the dynamics. Same stipulation for controlling task as mentioned in the caption of Fig. 4.

In summary, the method acts in two successive steps: a first recognition task in which the periods of UPOs are extracted from the unperturbed dynamics and a second control task whereby one can constrain the system to shadow a given periodic orbit. The second step can also be used for slaving a given dynamics $\boldsymbol{x}(t)$ to a goal dynamics $\boldsymbol{g}$ by simply redefining in Eq. (7) the differences as $\delta x_i(t_{n+1}) = x_i(t_{n+1}) - g_i(t_{n+1})$ and keeping Eqs. (8)–(10) with the new δ's for the updating τ process and for the controlling process.

The main difference between this method and OGY is that the adaptive technique does not act on the control parameters, and it needs neither to calculate the PS for the system under study, nor to study the local dynamics of different UPOs on the PS.

Finally, if one compare this control technique with that proposed in Ref. [11], two novel features characterize our method. First, the adaptive nature of the forcing term (Eq. (9)) which is inversely proportional to the time intervals and hence is weighted by the information extracted from the dynamics itself. Secondly, while in Ref. [11] the control is readjusted at each computational time step, here interventions are done at the intermediate time scale, thus reducing the computational or experimental effort for the control task.

Application of the adaptive algorithm to experimental data series embedded in spaces of increasing dimensions is discussed in Ref. [2]. Experimental application of such a method is in progress and will be reported elsewhere.

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References

[1] E.R. Hunt, Phys. Rev. Lett. 67 (1991) 1953.
[2] F.T. Arecchi, G. Basti, S. Boccaletti and A.L. Perrone, Europhys. Lett. 26 (1994) 327.
[3] S. Boccaletti and F.T. Arecchi, Europhys. Lett. 31 (1995) 127.
[4] D. Auerbach, P. Cvitanovic, J.-P. Eckmann, G. Gunarantne and I. Procaccia, Phys. Rev. Lett. 58 (1987) 2387.
[5] E. Ott, C. Grebogi and J.A. Yorke, Phys. Rev. Lett. 64 (1990) 1196.
[6] T. Shinbrot, C. Grebogi, E. Ott and J.A. Yorke, Nature 363 (1993) 411.
[7] B.B. Plapp and A.W. Huebler, Phys. Rev. Lett. 65 (1990) 2302; E.A. Jackson and A.W. Huebler, Physica D 44 (1990) 407; E.A. Jackson, Phys. Rev. A 44 (1991) 4839.
[8] R. Lima and M. Pettini, Phys. Rev. A 41 (1990) 726; Y. Braiman and I. Goldhirsch, Phys. Rev. Lett. 66 (1991) 2545.
[9] A. Azevedo and S.M. Rezende, Phys. Rev. Lett 66 (1991) 1342.
[10] H. Herzel, A. Angew, Math. Mech. 68 (1988) 11; S. Fahy and D.R. Hamann, Phys. Rev. Lett. 69 (1992) 761.
[11] K. Pyragas, Phys. Lett. A 170 (1992) 421.
[12] J. Singer, Y.-Z. Wang and H.H. Bau, Phys. Rev. Lett. 66 (1991) 1123.
[13] R. Roy, T.W. Murphy Jr., T.D. Maier, Z. Gills and E.R. Hunt, Phys. Rev. Lett. 68 (1992) 1259.
[14] B. Peng, V. Petrov and K. Showalter, J. Phys. Chem. 95 (1991) 4957; V. Petrov, V. Gaspar, J. Masere and K. Showalter, Nature 361 (1993) 240.
[15] R. Meucci, W. Gadomski, M. Ciofini and F.T. Arecchi, Phys. Rev. E 49 (1994) R2528.
[16] E.J. Romeiras, C. Grebogi, E. Ott and W.P. Dayawansa, Physica D 58 (1992) 165.
[17] E.J. Kostelich, C. Grebogi, E. Ott and J.A. Yorke, Phys. Rev. E 47 (1993) 305.
[18] S. Grossberg, Neural Network 1 (1988) 17.
[19] G. Basti and A.L. Perrone, Proc. Int. Symp. of Information Physics (ISKIT'92) Kyushu Inst. of Technology, July 12–15 (1992) p.122.
[20] E.N. Lorenz, J. Atmos. Sci. 20 (1963) 130.
[21] O.E. Roessler, Phys. Lett. 71 A (1979) 155.
[22] P. Cvitanovic, G.H. Gunaratne and I. Procaccia, Phys. Rev. A 38 (1988) 1503; G.H. Gunaratne, P.S. Linsay and M.J. Vinson, Phys. Rev. Lett 63 (1989) 1; G.B. Mindlin, X.-J. Hou, H.G. Solari, R. Gilmore and N.B. Tufillaro, Phys. Rev. Lett. 64 (1990) 2350; N.B. Tufillaro, H.G. Solari and R. Gilmore, Phys. Rev. A 41 (1990) 5717.