

Autonomous Bursting in a Homoclinic System

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The output of a dynamical system in a regime of homoclinic chaos transforms from a continuous train of irregularly spaced spikes to clusters of regularly spaced spikes with quiescent periods in between (bursting), provided a low frequency portion of the output is fed back. We provide experimental evidence of such an autonomous bursting by a CO₂ laser with feedback. The phenomena here presented are extremely robust against noise and display qualitative analogies with bursting phenomena in neurons.

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Spiking occurs in many physical and biological systems [1]. In neurons it is associated with the generation of action potentials [2]; in general, spikes are irregularly spaced and the precise time position of each spike within a train is considered to code relevant information [3]. A neuron can also display a different firing pattern called *bursting*, consisting of short trains of rapid spike oscillations intercalated by quiescent intervals, which periodically repeat. Bursting requires a neuron dynamics with two different time scales as [4]

$$\begin{aligned}\dot{X} &= F(X, Y) \\ \dot{Y} &= G(X, Y).\end{aligned}\quad (1)$$

Here, the vector X describes the fast variables generating spikes and Y the slow ones contributing a modulation. The fast variables are associated with the membrane potential, Na⁺ and K⁺ conductance and other variables evolving on time scales comparable with the duration of the spike, whereas the slow variables refer to the concentration and conductance of Ca²⁺ ions [5]. Bursting arises as the slow variables switch the fast ones between steady state and oscillatory dynamics. Thus a feedback from Y confers to X the feature of a bursting pattern consisting of a fast spiking regime riding on a slow modulation [5,6]. On the other hand, spiking phenomena are observed and easily modeled in other physical areas. Spike emission occurs in multimode class B lasers (solid state and semiconductor) operating close to threshold. This type of emission does not occur in a single-mode class B laser as CO₂, but it can be induced by a suitable feedback [7].

Here we present evidence of the autonomous transition from a continuous train of irregular spikes to periodic bursts of regularly spaced spikes in a laser system. We take as the fast system a laser undergoing homoclinic chaos [7]. We consider such a behavior as generic for all spiking systems based on an activator-inhibitor competition. Indeed the close approach to a saddle point, peculiar of homoclinic chaos, provides a local slowing down of the few variables (three-dimensional subset of X) describing the inflow to, and the outflow from, that unstable region. This means that all the other X variables do not play a crucial role around the singularity, whence such a generic homoclinic

behavior applies to many different models which have in common the presence of a saddle point. Even though the core features around the saddle require three coupled variables, we will use a six-dimensional model in order to recover the global aspects of the experimental dynamics. In this system, the presence of *nonautonomous* bursting induced by an external modulation has already been reported [8]. Motivated by the fact that biological bursting *does not* require an *external forcing*, we explore the feasibility of *autonomous* bursting in the laser. This is achieved by a secondary feedback loop, where a fraction of the output is fed back to a loss modulator after passage through a low pass filter. The slow dynamics thus provides the mechanism for bursting the laser intensity.

The experiment has been performed on a single mode CO₂ laser with an intracavity loss modulator; the detected output intensity is fed back in order to control the cavity losses (Fig. 1). The dc level and amplitude of the feedback signal are controlled by adjusting the bias B and the gain R of the amplifier. We set these two parameters so that the laser is in a regime of Shilnikov chaos (the saddle point being a saddle focus [9]). The laser intensity, after a time interval where it practically approaches zero, displays a large spike followed by a damped train of fast oscillations and a successive train of growing oscillations (Fig. 2).

Damped and growing trains represent, respectively, the approach to, and the escape from, the saddle focus S from where the trajectory rapidly returns to zero and then starts

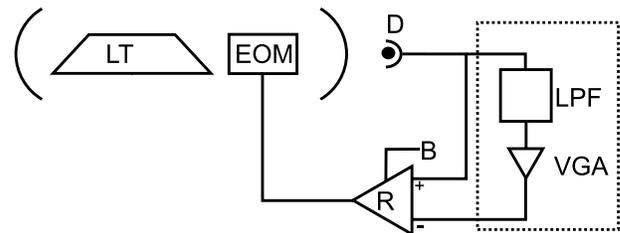


FIG. 1. Experimental setup consisting of a CO₂ laser with a primary feedback loop imposing a regime of homoclinic chaos and a secondary slow feedback loop (dashed block) which produces bursting. LT, laser tube; EOM, electro-optic modulator; D, HgCdTe detector; LPF, low pass filter; VGA variable gain amplifier (ac coupled).

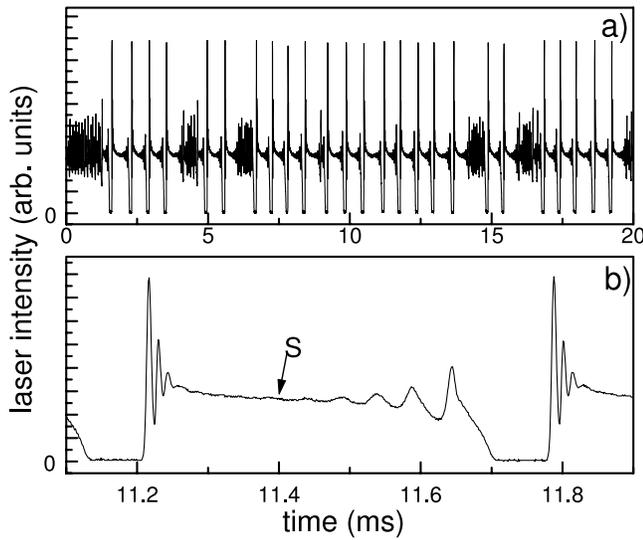


FIG. 2. (a) Experimental time series of the laser output intensity in the homoclinic chaos regime (zero intensity level indicated). (b) Zoom of a single pulse showing the approximate location of the saddle focus S at the start of the growing oscillation; notice that the preceding large damped oscillation occurs away from S and it is not part of the local dynamics at S .

a new orbit. In the chaotic region, the escape time from the saddle focus has a different duration from pulse to pulse. The power spectrum shows a broadened peak at the average repetition frequency of the pulses, $\nu_n \approx 2$ kHz. The non vanishing frequency content below ν_n suggests that it is possible to enhance one of those frequencies by feeding back a filtered fraction of the output. Thus, we filter the detected signal by a low pass filter with a cutoff frequency (ν_c) lower than ν_n and then reinject it through a secondary feedback loop. As a result, taking also into account the phase response of the filter, we select a bursting frequency somewhat lower than, but proportional to, ν_c . The other low ($\nu < \nu_c$) frequency components are efficiently suppressed. In this way, we obtain clusters of laser pulses periodically spaced at the bursting frequency. We have used “Rockland model 452” filters with a slope of 40 dB/decade. It was experimentally tested that a single filter is not sufficient; we were obliged to cascade two filters. In fact, we will see later that the numerical model requires at least three poles. The secondary feedback loop contains also a variable gain amplifier, ac coupled in order not to alter the primary feedback loop setting. The filter yields bursting sequences with adjustable periods, reset by changing ν_c (Fig. 3).

The low frequency feedback corresponds to coupling two independent lasers, even though we have done the experiment on a single laser system, hence the phenomenon here investigated has a cooperative character. Indeed, the correlation time of the chaotic signal is measured to be around 100 μ s, consistently lower than the average interspike interval ($\nu_n^{-1} \approx 500$ μ s). Once we select a feedback

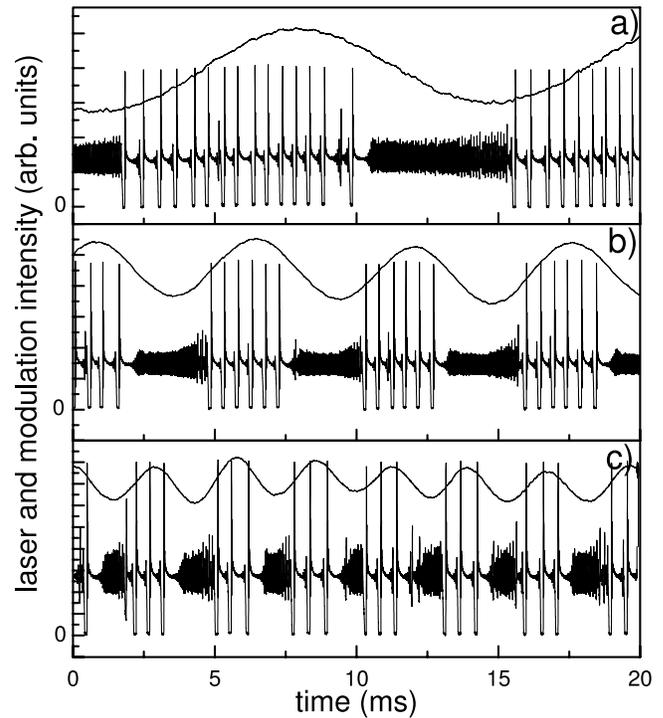


FIG. 3. Experimental time series of bursting obtained with the Rockland filter for three different values of the cutoff frequency: (a) $\nu_c = 100$ Hz, (b) $\nu_c = 300$ Hz, (c) $\nu_c = 600$ Hz. The low frequency sinusoids are trains of the filtered feedback signal reported on a different scale; in fact the feedback amplitude is 2% of the spike amplitude. The effective bursting frequencies are, respectively, (a) 80 Hz, (b) 200 Hz, and (c) 350 Hz; that is, proportional to, even though lower than, the corresponding cutoff ν_c .

frequency around 500 Hz, well below ν_n , the feedback onset time (≈ 2 ms) is much longer than the decorrelation time, thus our feedback is equivalent to the cooperative interaction of independent lasers.

The dynamics is modeled by coupling the laser, described by a six-dimensional state vector $X \{x_i; i = 1 - 6\}$ [10], to a three-dimensional filter $Y \{y_i; i = 1 - 3; y_0 \equiv x_1$ is the filter input):

$$\begin{aligned}
 \dot{x}_1 &= k_0 x_1 [x_2 - 1 - k_1 \sin^2(x_6)], \\
 \dot{x}_2 &= -\gamma_1 x_2 - 2k_0 x_2 x_1 + g x_3 + x_4 + p_0, \\
 \dot{x}_3 &= -\gamma_1 x_3 + x_5 + g x_2 + p_0, \\
 \dot{x}_4 &= -\gamma_2 x_4 + g x_5 + z x_2 + z p_0, \\
 \dot{x}_5 &= -\gamma_2 x_5 + z x_3 + g x_4 + z p_0, \\
 \dot{x}_6 &= -\beta \{x_6 + B - R[x_1 - \phi(y_3 - \bar{y}_3)]\}, \\
 \dot{y}_i &= -\theta(y_i - y_{i-1}).
 \end{aligned} \tag{2}$$

In these equations, x_1 is the normalized laser intensity, x_2 and x_3 are the population difference and sum of the two molecular levels resonant with the field, x_4 and x_5 are the difference and sum of the populations of the two

TABLE I. Parameter values used in the simulations.

γ_1	10.0643	k_0	28.5714
γ_2	1.0643	k_1	4.556
z	10	p_0	0.016
β	0.4286	R	160
g	0.05	B	0.1031
θ	$[4.5 \times 10^{-4} - 2.7 \times 10^{-3}]$	ϕ	4

rotational manifolds, which exchange energy with the two resonant levels, x_6 is the feedback voltage, γ_1 , γ_2 , g , and β are decay rates, p_0 is the pump, and z is the number of sublevels contained in the rotational manifolds. In the x_6 equation ϕ is the gain of the secondary feedback amplifier. The time is rescaled according to $t = 7 \times 10^5 t$.

The slow dynamics corresponds to a low pass filter with a third order pole at $\theta = 2\pi\nu_c$, modeled by the cascade of three linear integrators. The numerical values of the parameters have been chosen so that the free running behavior of the laser displays Shilnikov chaos [10] (Table I). The filter variables represent the slow subsystem Y of Eqs. (1); the Y equations are similar to those used in the Hindmarsh and Rose model for neuronal bursting [11].

Figure 4 shows the model behavior at the onset of the stable bursting corresponding to a perturbation of 2%. It is important to note that the use of a single pole filter does not provide bursting because the frequency component corresponding to the average repetition frequency of the chaotic pulses ($\nu_n \approx 2$ kHz) is not sufficiently rejected. In this

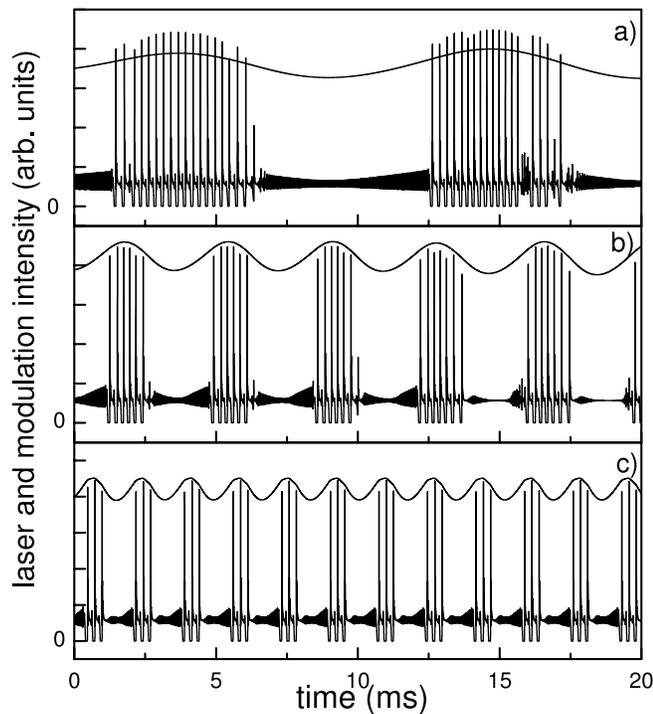


FIG. 4. Numerical results corresponding to the three cases reported in Fig. 3. (a) $\theta = 4.5 \times 10^{-4}$ ($\nu_c = 100$ Hz); (b) $\theta = 1.3 \times 10^{-3}$ ($\nu_c = 300$ Hz); (c) $\theta = 2.7 \times 10^{-3}$ ($\nu_c = 600$ Hz).

case, the action of the secondary feedback results in a modulation at the average repetition frequency.

In our case bursting is related to homoclinic chaos, which requires a fine parameter tuning in order to set the orbits around the saddle focus [8–10]. As the control parameter B is moved above or below the homoclinic value, the dynamical system enters, respectively, a periodic behavior or it goes to a metastable fixed point (excitable system). This is clearly shown by a $\pm 1\%$ stepwise change of B (Figs. 5a and 5b). Furthermore, in the case of a positive perturbation, the repetition frequency ν_r of the periodic spike train is a monotonic function of the perturbation amplitude, up to 5% (Fig. 5c).

The above scenario proves a fact relevant for neurodynamics, that *noise is not necessary* to induce bursting. We have checked the independence of autonomous bursting from noise by applying to the laser pump an additive white noise up to 10% of the pump's nominal value, without observing modifications. This demonstrates that autonomous bursting *does not belong* to the class of noise induced phenomena as, e.g., stochastic resonance [12] or coherence resonance [13].

To make comparison with other areas, this behavior is found in many excitable biological systems as well as in

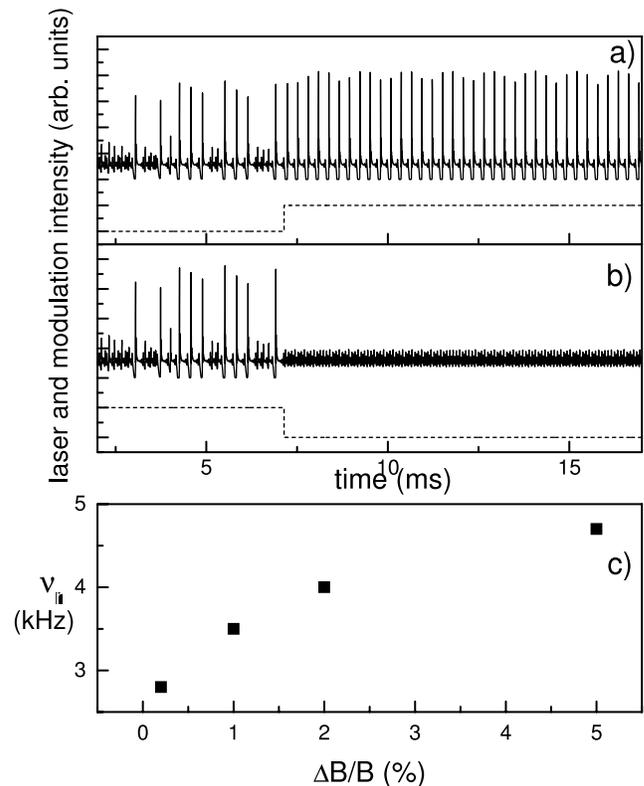


FIG. 5. Stepwise increase (a) and decrease (b) of control parameter B by $\pm 1\%$ (dashed line) brings the system from homoclinic behavior to, respectively, periodic or excitable behavior. (c) In the case (a) the frequency repetition ν_r of the spikes within the regular train increases monotonically up to a $\Delta B/B \sim 5\%$; above this value, there is a saturation.

chemical reactions [14–16]. In the neural context bursting phenomena have been found in different cell types; as thalamic neurons [17], hippocampal pyramidal neurons [18], and pancreatic β cells [19,20]. Some real neurons do not show bursting when the synaptic connections with other cells are blocked. In such a condition, a single neuron typically shows chaotic spiking activity. Examples are provided by pancreatic β cells [21] or by neurons of the pyloric central pattern generator in the stomatogastric ganglion of crustaceans [22]. In these cases the appearance of bursting is related to the synaptic interaction among the cells [23–25]. This behavior can be interpreted in the framework of our model if we take the secondary feedback term as inducing a cooperative coupling among identical coupled lasers. As for the frequency ν_r of the regular spiking within each burst, Fig. 5c shows that it depends sensitively upon the B perturbation. This suggests that an accurate control of calcium concentration might be the neurodynamic tool for a reliable and sensitive coding of different stimuli, without need to recur to noise [3,26].

In conclusion, the chaotic spiking regime peculiar of homoclinic chaos has been converted to an *autonomous bursting* regime by means of a feedback loop acting on a long temporal scale as compared to the average repetition time of the chaotic pulses. The observed regimes display characteristics similar to those observed in neurons [5,17,18] and are reproduced by a model where the fast dynamics is modulated by a slow one through a secondary feedback loop; the frequency of the modulation sets the bursting period, while its amplitude sets the spike repetition frequency within each burst.

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