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EFFECTS OF NOISE ON HOMOCLINIC CHAOS LaTeX

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The effect of noise on the dynamics of an homoclinic chaotic system is investigated. Homoclinic chaos is characterized by strong fluctuation in the time intervals between sucessives regular spikes, such a dynamics is in opposition with the phase coherent dynamics of most common chaoitc systems which are characterized by chaotic fluctuation in the amplitude but not in the phase. We demonstrate,both numerically and experimentally in a CO_2 laser, different effects of noise on the homoclinic chaotic dynamics.

Keywords: homoclinic chaos; synchronization.

1. Introduction

Resonant response of a nonlinear system to a weak driving signal has been investigated in various contexts. In self-sustained periodic oscillators, the system adjusts its time scale, achieving frequency and phase locking to the driving signal. Such a phenomenon, characterized by an Arnold tongue synchronization region in the parameter space of the amplitude and frequency of the driving signal, is of fundamental importance with applications in various fields [1].

Recently the study of phase synchronization (PS) has been extended to chaotic oscillators [2–4] and to chemical [5], laser [6] and other [7] experimental systems whith chaotic dynamics. The most studies refer to systems which displays very coherent phase dynamics due to a small fluctuation of the return time T between two successive crossing of a Poincaré section although the amplitudes fluctuate strongly [4]. The effect of a periodic weack modulation [2, 4] or a weack coupling between two non identical chaotic oscillators can produce PS [3], the amplitudes remaining chaotic. The use of a strong coupling induce a complete synchronization, both phase and amplitude are synchronized [8].

Noise can have different effects on the synchronization phenomenon. Generallyh it induces phase slips and smeares out the border of the synchronization region [9]. However noise can have also a constructive effect such as stochastic resonance (SR) [10] or coherence resonance (CR) [11]. These phenomena, mainly studied on bistable or excitable systems refere to the coherent response that a suitable amount of noise can induce. For SR the choerent response is to an external subtreshold forcing while in CR noise alone can generate the most coherent motion without an external signal.

Other effect of noise on dynamical systems which has been subjet of a long controversie is the possibility of induce a synchronized regime in two identical system not coupled but driven by a common noise signal [15–17]. It has been shown that a nonzero mean (biased) noise plays a decisive role by shifting the dynamics toward a stable regime [16, 17]. However, a general conclusion [17] that an unbiased noise cannot lead to synchronization has been disproved by recent examples [18]. This long-standing controversy has been recently clarified by showing that the key mechanism of noise induced synchronization (NIS) is the existence of a large *contraction region* in the phase space where nearby trajectories converge to each other [19]. The effect of noise is to change the competition between contraction and expansion, and synchronization occurs if the contraction becomes dominant.

A class of self-sustained chaotic oscillations which exhibit quite different behavior as compared to phase coherent chaotic oscillations is . hmoclinic chaos [20]. Typically, these chaotic oscillators possess the structure of a saddle point S embedded in the chaotic attractor. The chaotic trajectories starting from a neighborhood of S leave S slowly along the unstable manifold and have a fast and close recurrence to S along the stable manifold after a large excursion (spike). Thus a significant contraction region exists close to the stable manifold. The dynamics is characterized by a sequence of spikes with widely fluctuating inter spike interval ISI. Such a structure underlies spiking behavior in many neuron [21], chemical [22] and laser [23] systems.

In this letter we investigate noise effects on homoclinic chaos. We find that weak noise can enhance the coherence of the spike train. As a result, the largest Lyapunov exponent (LLE) becomes negative and a weak common noise can induce complete synchronization of identical uncoupled systems. This noise induced changes also enhances greatly the response of the system to a weak signal. The noisy systems display resonances with respect to both the noise intensity and the signal frequency.

2. Setup

We demonstrate these nontrivial effects of noise in a single mode CO_2 laser, both numerically and experimentally. The experimental setup consists of a CO_2 laser with an intracavity loss modulator, driven by a feedback signal which is proportional to the laser output intensity. The system is operating in a homoclinic chaos regime where the laser output consists of a chaotic sequence of spikes [23,24]. To investigate the role of external noise a the signal of a Gaussian noise generator is added to the feedback loop. The noise generator has a high frequency cut-off at 50 kHz, but for all purposes it can be regarded as a white noise source. For the numerical simulation the followin model has been used

$$\dot{x}_1 = k_0 x_1 (x_2 - 1 - k_1 \sin^2 x_6), \quad (1)$$
$$\dot{x}_2 = -\gamma_1 x_2 - 2k_0 x_1 x_2 + g x_3 + x_4 + p_0, \quad (2)$$
$$\dot{x}_3 = -\gamma_1 x_3 + g x_2 + x_5 + p_0, \quad (3)$$
$$\dot{x}_4 = -\gamma_2 x_4 + z x_2 + g x_5 + z p_0, \quad (4)$$
$$\dot{x}_5 = -\gamma_2 x_5 + z x_3 + g x_4 + z p_0, \quad (5)$$
$$\dot{x}_6 = -\beta(x_6 - b_0 + \frac{r x_1}{1 + \alpha x_1}) + D\xi(t). \quad (6)$$

Here, x_1 represents the laser output intensity, x_2 the population inversion between the two resonant levels, x_6 the feedback voltage signal which controls the cavity losses, while x_3, x_4 and x_5 account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band. Furthermore, k_0 is the unperturbed cavity loss parameter, k_1 determines the modulation strength, g is a coupling constant, γ_1, γ_2 are population relaxation rates, p_0 is the pump parameter, z accounts for the number of rotational levels, and β, r, α are respectively the bandwidth, the amplification and the saturation factors of the feedback loop. With the following parameters $k_0 = 28.5714$, $k_1 = 4.5556$, $\gamma_1 = 10.0643$, $\gamma_2 = 1.0643$, $g = 0.05$, $p_0 = 0.016$, $z = 10$, $\beta = 0.4286$, $\alpha = 32.8767$, $r = 160$, and $b_0 = 0.1031$, the model reproduces the regime of homoclinic chaos observed experimentally [24]. The model has been implemented in order to take into account also the small intrinsic noise of the system which has been evaluated as 0.14% ($D = 0.0005$) perturbation of the feedback signal x_6. The model is integrated using a Heun algorithm [25] with a small time step $\Delta t = 5 \times 10^{-5}$ ms (note that typical $T \sim 0.5$ ms).

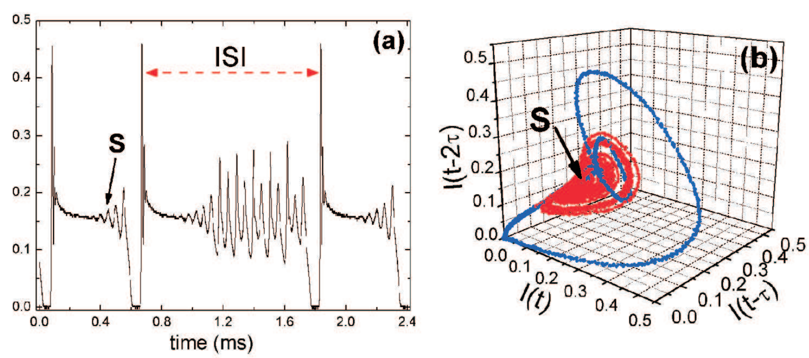

Fig 1. (a) Laser intensity and (b) reconstructed 3D attractor of the laser dynamcs in the regime of homoclinic chaos showing the exstence of the expation (red points) and contraction (blue points) regions representing the escape and approach from the seddle focus S respectively. The existence of the saddle focus induce large fluctuation on the inter spike intervals (ISI).

Laser output displays large spikes, followed by a fast damped train of a few oscillations towards S and a successive longer train of growing oscillations spiraling out from S (Fig. 1a). The damped oscillation manifests strong contraction along the stable manifold in the phase space, while the growing one manifests weak expansion along the unstable manifold (Fig. 1b). The dynamics of the laser in such a condition is characterized by two well distinct region where the trajectory are contracted and expanded respectively Fig.1. For this reazon noise can play an interesting role on the dynamics of the system.

3. Noise effects

The model of the laser dynamics without considering the intrinsic noise displays a broad range of time scales. There are many peaks in the distribution $P(ISI)$ of ISI, as shown in Fig. 2a. In the presence of noise, the trajectory on average cannot come closer to S than the noise level and the system spends a shorter time following the guidance of the unstable manifold. As a consequence small noise ($D = 0.0005$) changes the time scales of the model: $P(ISI)$ is now characterized by a dominant peak followed by a few exponentially decaying ones (Fig. 2b).

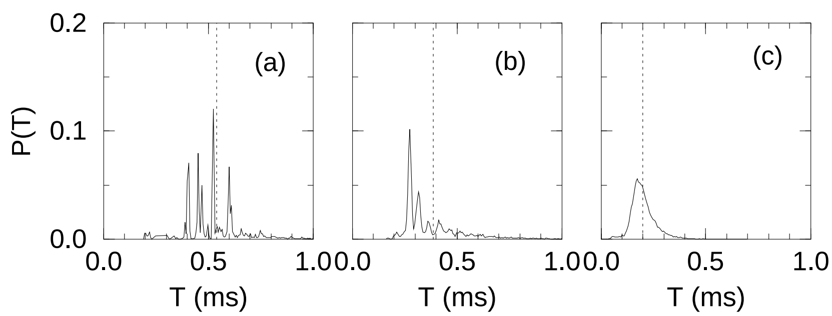

Fig 2. Noise-induced changes in time-scales. (a) $D = 0$; (b) $D = 0.0005$; and (c) Experimental results corresponding to the simulation with only intrinsic noise (a). The dotted lines shows the mean interspike interval $T_0(D)$, which decreases with increasing D.

This distribution of ISI is typical for small D in the range $D = 0.00005 \div 0.002$. The experimental system with only intrinsic noise (equivalent to $D = 0.0005$ in the model) has a very similar distribution $P(ISI)$ of that reported in Fig. 2c. At larger noise intensity $D = 0.01$, the fine structure of the peaks is smeared out and $P(ISI)$ becomes an unimodal peak in a smaller range. In addition the noise change the mean value $ISI_0(D) = \langle ISI \rangle_t$ (Fig. 2, dotted lines) that decreases with D. When the noise is rather large, it affects the dynamics not only close to S but also during the spiking, so that the spike sequence becomes fairly noisy.

We observe the most coherent spike sequences at a certain intermediate noise. We quantify the coherence by R [11]

$$R = ISI_0(D)/\sigma_{ISI}, \tag{7}$$

where σ_{ISI} is the standard deviation of $P(ISI)$. When D increases, R reaches a maximal value and decreases again (Fig. 3a,b), exhibiting CR [1], both in the model and experimental systems.

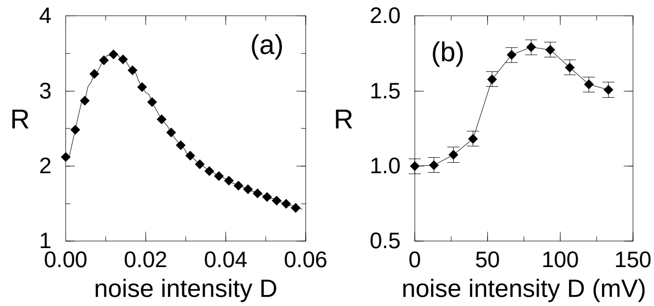

Fig 3. Coherence resonance (CR) in the model (a) and experimental (b) systems.

The noise has an effect in the competition between contraction and expansion; for the reduced time spent by the system in the neighbhoord of the saddle point the contraction may become dominant. To measure this changes, we calculate the LLE λ_1 in the model as a function of the noise intensity D (Fig. 4a, dotted blue line). λ_1 undergoes a transition from a positive to a negative value at $D_c \approx 0.0031$. Beyond D_c, two identical laser models $\mathbf{x}$ and $\mathbf{y}$ with different initial conditions but the same noisy driving $D\xi(t)$ achieve complete synchronization, as shown by the vanishing normalized synchronization error $E = \frac{\langle |x_1 - y_1| \rangle}{\langle |x_1 - \langle x_1 \rangle| \rangle}$ (Fig. 4a, solid green line).

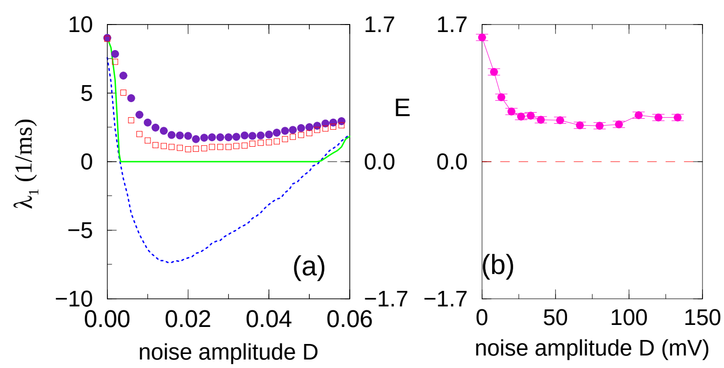

Fig 4. Noise induced synchronization (NIS) in the model (a) and experimental (b) systems. (a) Dotted blue line: the LLE λ_1, solid green line: normalized synchronization error E between two fully identical laser models $\mathbf{x}$ and $\mathbf{y}$, closed blue circles: E between two lasers with a small independent noise (intensity $D_1 = 0.0005$) and open red squares: E between two nonidentical lasers with $b_0 = 0.1031$ and $b_0 = 0.1032$, vs intensity D of the common noise.

At larger noise intensities, expansion becomes again significant, and the LLE increases and synchronization is lost when λ_1 becomes positive for $D > 0.052$. By considering the small intrinsic noise source present in the experimental system in the model the normalized synchronization error decreas as a function of noise but do not reach the zero. In experimental study of NIS, for each noise intensity D we repeat the experiment twice with the same external noise. As consistent with numerical results with small independent noise, E does not reach zero due to the intrinsic noise, and it increases slightly at large D (Fig. 4b).

Our results disprove experimentally previous claim that NIS can be achieved only by noise with a nonzero mean value [16, 17], and verify that a significant contraction region, which exists generically in homoclinic chaotic systems due to the stable manifold of the saddle S, plays the decisive role [19].

By applying a small perturbation to the pump paramenter we investigate how the noise-induced changes in the time scale affect the response of the system to a weak external driving signal.

We investigated systematically the response sensitivity of the laser model by analyzing the synchronization for a driving amplitude $A = 0.01$ and the relative initial frequency difference $\Delta\omega = (f_e - f_0(D))/f_0(D)$, where the average frequency $f_0(D)$ of the unforced laser model is an increasing function of D. The actual relative frequency difference in the presence of the signal is calculated as $\Delta\Omega = (f - f_e)/f_0(D))$, where $f = 1/\langle ISI \rangle_t$ is the average spiking frequency of the forced laser model.

We show in Fig. 5(a) that the system exhibit a very complicated and unusual response to a weak driving signal in the noise-free model, however thus has not been observed in the experimental system [26] due to the intrinsic noise whose intensity is equivalent to $D = 0.0005$ in the model. As reported in Fig. 5 synchronization both in the model and in the experiment can be enhanced by added external noise, especially for $\Delta\omega > 0$; an external noise too strong degrades synchronization again (Fig.5).

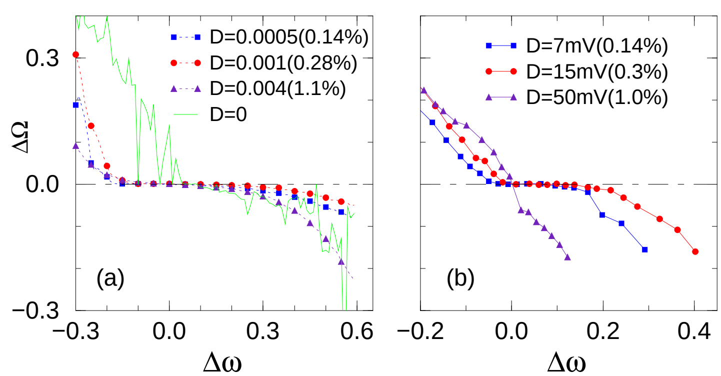

Fig 5. Noise-enhanced PS: a comparison between model and experimental systems. (a) model, $A = 0.01$. (b) experiment: signal amplitude 10mV ($A = 0.01$); the noise intensity denotes total noise measured in the feedback loop, and $D = 7$mV corresponds to the intrinsic noise. In both cases, the noise intensities are also indicated in % of the feedback signal x_6.

Thus, noise can play a constructive role to enhance frequency locking and PS of homoclinic chaos to a weak driving signal. Without noise, the model system exhibits a very complicated and unusual response to the signal; whereas in the presence of a small noise, it displays a resonance versus the frequency, as in usual phase coherent oscillators.

Furthermore the PS behavior is optimized at a certain noise intensity, as SR [10, 12, 13]. In the following, we study in detail how this SR is affected by noise intensity

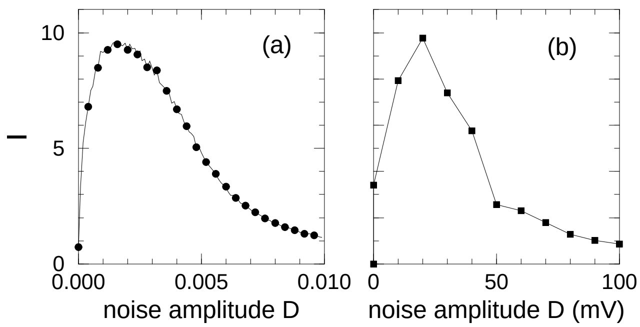

Fig 6. Stochastic resonance for a fixed driving period. Left panel: model, $A = 0.01, T_e = 0.3$ms. Right panel: experiments: forcing amplitude 10mV ($A = 0.01$) and period $T_e = 1.12$ms.

D.

We now study how the response is affected by noise intensity D for fixed signal period T_e. Here, in the unforced homoclinic chaotic lasers the average interspike interval $ISI_0(D)$ decreases with increasing D, and SR similar to that in bistable or excitable can also be observed. We employ the following measure of coherence as an indicator of stochastic resonance [14]

$$I = \frac{T_e}{\sigma_{ISI}} \int_{(1-\alpha)T_e}^{(1+\alpha)T_e} P(ISI)dISI, \tag{8}$$

where and $0 < \alpha < 0.25$ is a free parameter. This indicator takes into account both the fraction of spikes with an interval roughly equal to the forcing period T_e and the jitter between spikes [14]. Evidence of SR to the driving signal has been demonstrated both in the model and in the experimental systems by I as reported in Fig.6.

4. Conclusion

In summary, we have shown that in homoclinic chaotic systems characterized by a strong fluctuation of the interspike interval, the time scales become more regular with a small noise due to reduced residence time in the weak unstable region. Various constructive effects have been demonstrate both numerically and experimentally as a consequence of this noise-induced change in time scale. The system displays coherence resonance without an external signal and enhanced phase synchronization, deterministic resonance and stochastic resonance in response to a weak signal. Importantly, we have also shown that unbiased noise is sufficient to induce complete synchronization. Moreover, NIS is not necessarily associated with large noise intensities, but in the case of homoclinic chaos it can be induced by weak noise which does not significantly alter the geometry of the dynamics.

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References

[1] Y. Kuramoto, *Chemical Oscillations, Waves and Turbulence* (Springer, Berlin, 1984).

[2] E.F. Stone, Phys. Lett. A **163**, 367 (1992); A.S. Pikovsky *et al.*, Physica D **104**, 219 (1997).

[3] M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **76**, 1804 (1996).

[4] A.S. Pikovsky, M.G. Rosenblum, G.V. Osipov, and J. Kurths, Physica D **104**, 219 (1997).

[5] I.Z. Kiss and J.L. Hudson, Phys. Rev. E 2001, **64**, 046215 (2001); Phys. Chem. Chem. Phys. **4**, 2638 (2002); I.Z. Kiss, Y. Zhai and J. L. Hudson, Science **296**, 1676 (2002).

[6] D.Y. Tang *et al.*, Chaos 8, 697 (1998); D.J. Deshazer *et al.*, Phys. Rev. Lett. **87**, 044101 (2001); K.V. Volodchenko *et al.*, Opt. Lett. **26**, 1406 (2001); N. Kravtsov *et al.*, Quantum Electron. **32**, 251 (2002).

[7] D. Maza *et al.*, Phys. Rev. Lett. **85**, 5567 (2000); C.M. Ticos *et al.*, *ibid.* **85**, 2929 (2000).

[8] L.M. Pecora and T.L. Carroll, Phys. Rev. Lett. **64**, 821 (1990); M.G. Rosenblum, A.S. Pikovsky, and J. Kurths, Phys. Rev. Lett. **78**, 4193 (1997).

[9] R. L. Stratonovich, *Topics in the Theory of Random Noise* (Gordon and Breach, New York, 1967), Vol 2.

[10] R. Benzi, A. Sutera, and A. Vulpiani, J. Phys. A **14**, L453 (1981); K. Wiesenfeld and F. Moss, Nature **373**, 33 (1995); L. Gammaitoni, P. Hänggi, P. Jung, and F. Marchesoni, Rev. Mod. Phys. **70**, 223 (1998).

[11] G. Hu *et al.*, Phys. Rev. Lett. **71**, 807 (1993); A.S. Pikovsky and J. Kurths, *ibid.*, **78**, 775 (1997); G. Giacomelli, M. Giudici, S. Balle, and J.R. Tredicce, *ibid.*, **84**, 3298 (2000); C.S. Zhou, J. Kurths and B. Hu, *ibid.* **87**, 098101 (2001);

[12] L. Gammaitoni *et al.*, Phys. Rev. Lett. **74**, 1052 (1995); B. Shulgin, A. Neiman, and V. Anishchenko, Phys. Rev. Lett. **75**, 4157 (1995).

[13] A. Neiman *et al.*, Phys. Rev. E **58**, 7118 (1998).

[14] F. Marino *et al.*, Phys. Rev. Lett. **88**, 040601 (2002).

[15] A. Mritan and J.R. Banavar, Phys. Rev. Lett. **72**, 1451 (1994); A. Pikovsky, Phys. Rev. Lett. **73**, 2931 (1994); L. Longa, E.M.F. Curado and A.A. Oliveira, Phys. Rev. E **54**, R2201 (1996).

[16] H. Herzel, and J. Freund, Phys. Rev. E **52**, 3238 (1995); G. Malescio, *ibid.* **53**, 6551 (1996); P.M. Gade, and C. Basu, Phys. Lett. A **217**, 21 (1996).

[17] E. Sanchez, M.A. Matias and V. Perez-Munuzuri, Phys. Rev. E **56**, 4068 (1997).

[18] C.H. Lai, and C.S. Zhou, Europhys. Lett. **43**, 376 (1998); R. Toral, C.R. Mirasso, E. Hernandez-Garcia, and O. Piro, Chaos 11, 665 (2001); L. Baroni, R. Livi and A. Torcini, Phys. Rev. E, **63**, 036226 (2001).

[19] C.S. Zhou and J. Kurths, Phys. Rev. Lett. 88, 230602 (2002).

[20] L. P. Shilnikov, Math. USSR Sb. **10**, 91 (1970).

[21] A. L. Hodgkin and A. F. Huxley, J. Physiol. **117**, 500 (1952); U. Feudel *et al*, Chaos **10**, 231 (2000).

[22] F. Argoul, A. Arnéodo, and P. Richetti, Phys. Lett. **120A**, 269 (1987).

[23] F. T. Arecchi, R. Meucci, and W. Gadomski, Phys. Rev. Lett. **58**, 2205 (1987); F.T. Arecchi *et al.*, Europhys. Lett. **6**, 677 (1988).

[24] A.N. Pisarchik, R. Meucci and F.T. Arecchi, Eur. Phys. J. D **13**, 385 (2001).

[25] J. Garcia-Ojalvo and J.M. Sancho, *Noise in Spatially Extended Systems* (Springer, New York, 1999).

[26] E. Allaria, F.T. Arecchi, A. di Garbo and R. Meucci, Phys. Rev. Lett. **86**, 791 (2001).