Controlling transient dynamics to communicate with homoclinic chaos

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A control that stabilizes the transient dynamics of a homoclinic chaotic laser is used to encode discrete sources of information. The controlled trajectory is a complex spiking signal that has a constrained interspike interval, and therefore, the ratio of information transmitted is approximately constant. We also show that the controlled signal that encodes the source contains more information than the source. This property is advantageously used to correct possible errors in the transmission, or to increase the ratio of information per transmitted spike.
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Communication means the exchange of information between a transmitter and a receiver. The information is encoded on a carrier signal at the transmitter, and then decoded upon arrival at the receiver. The signal has to be robust against the limitations imposed by the channel. In this paper we consider limitations due to dropouts, that is, to interruptions in the transmission of information. Thus, the transmitted information should be recovered using only part of the signal. We show that a chaotic laser can be easily controlled such that its trajectory cannot only encode a random sequence of zeros and ones, but also the memory associated with the controlled trajectory can be used to partially recover the transmitted information, when the receiver only partially knows the transmitted signal.

Chaotic dynamical systems have deterministic properties, whereby given a short term evolution out of an initial condition, there are ways to retrieve that initial condition, or vice versa, i.e., given an initial condition its future evolution can be *predicted*. Chaotic systems also display an infinite number of periodic orbits, as well as *sensitivity to initial conditions* that makes a trajectory to approach arbitrarily close to all these periodic orbits, in a *recurrent* way. These properties were exploited in Refs. 1–4, which describe ways to encode a source of information in chaotic orbits, so that, using the predictive property of chaotic systems, one can recover lost information when dropouts occur. The recurrent property assures that an orbit can always be found that encodes some information that is smoothly connected to another orbit that encodes some other information.

However, finding these smooth orbits and controlling the system in order to achieve this final orbit is not a trivial task in experimental chaotic systems.[2] In addition, the implementation of a technique that explores chaotic characteristics in order to recover lost information is difficult to realize considering the complexity of such task. Finally, the large frequency bandwidth present in a chaotic signal, although useful to create a signal robust to noise,[5] can be seen as a disadvantage insofar as the time one would have to use the channel, in order to transmit some constant amount of information, would vary according to the particular encoding trajectory. So, the ratio of information, amount of information transmitted per channel use, when using encoding chaotic trajectories might not be constant, which brings technological difficulties for a chaos-based communication implementation.

In the past, several techniques for exploiting chaotic dynamics to communicate were implemented that make use of chaos control[6] and synchronization.[7] The purpose of the present work is to show how a simple method of control, first introduced in connection to a CO_2 laser with feedback,[8] stabilizes the transient dynamics of a spiking homoclinic chaotic laser.[9] The controlled orbit is recurrent to itself and its interspike interval is approximately constant. Therefore, the ratio of information transmission is independent of the encoding trajectory. In addition, we show that the proposed scheme offers a simple way of choosing the perturbations in order to achieve the desired encoding orbit (encoding), as well as to recover the information from these trajectories (decoding). In this work we assume that the spiking signal can be identified even when noise is added to it. Therefore, information encoding in such a system is robust against applied noise, provided that the noise amplitude is smaller than the spiking height.

The experimental setup of the single-mode CO_2 laser showing homoclinic chaos can be seen in Ref. 9. The dynamical equations used to simulate it are

$$\begin{aligned}
\dot{x}_1 &= k_0 x_1[x_2 - 1 - k_1 \sin(x_6)^2],\\
\dot{x}_2 &= -\gamma_1 x_2 - 2k_0 x_2 x_1 + g x_3 + x_4 + p_0,\\
\dot{x}_3 &= -\gamma_1 x_3 + x_5 + g x_2 + p_0,\\
\dot{x}_4 &= -\gamma_2 x_4 + g x_5 + z(x_2 + p_0),\\
\dot{x}_5 &= -\gamma_2 x_5 + z x_3 + g x_4 + z p_0,\\
\dot{x}_6 &= -\beta\left(x_6 + b_0 - \frac{r x_1}{1+\alpha x_1}\right) - \epsilon\theta f(t) + D\xi,
\end{aligned} \tag{1}$$

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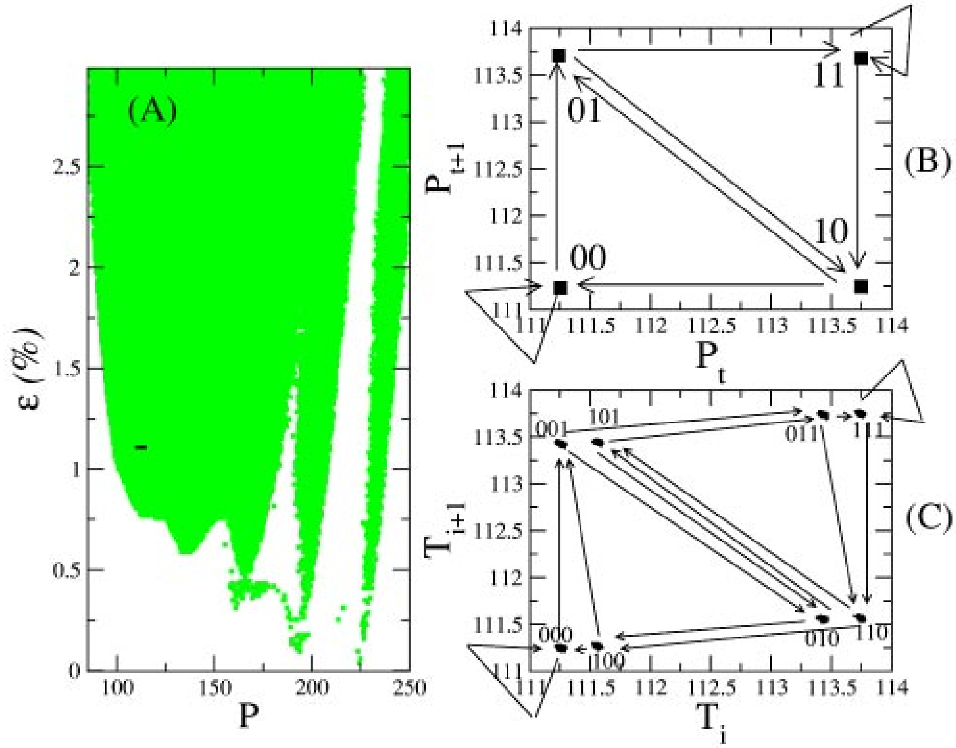

FIG. 1. (Color online) (a) Gray corresponds to regions where there is a period-1 attractor, for noise amplitude of $D=0.000\,15$. The horizontal segment indicates the position of the interval I. (b) The input space and permitted transitions of the source message. (c) The output map of the returning time to $x_1=0.003$, and the permitted transitions of the trajectory T, for noise amplitude $D=0.0001$, and pulse amplitude of $\epsilon=1.1\%$.

where $k_0=28.5714$, $k_1=4.5556$, $\gamma_1=10.0643$, $g=0.05$, $p_0=0.016$, $\gamma_2=1.0643$, $z=10$, $\beta=0.4286$, $b_0=0.1031$, $r=160$, $\alpha=32.8767$. $f(t)$ represents a pulse signal with amplitude 1, that is, $f(t)=1$ for $t=[mP,mP+\delta]$, where the pulse length $\delta=25.23$ (10% the average interspike period for a chaotic laser), and we call the time period of the pulse signal by P. The pulse is a negative perturbation that forces the laser to escape from the vicinity of a tangled chaotic region. The term $D\xi$ represents an independent Gaussian noise variable, with zero mean, and standard deviation $\sqrt{D}$, used to mimic internal stochastic noise present in the experimental system. The amplitude of the pulse is represented by $\epsilon\theta$, with $\theta=\beta r*0.005$, where 0.005 is approximately the maximum amplitude of the variable x_1. Examples of the laser intensity signal as a representation of the pulsed perturbations can be seen in Ref. 8.

To give the orders of magnitude, the spike train which characterizes the laser behavior consists of a sequence of identical pulses, each one lasting about 10 μs, and separated from each other by an interspike interval (ISI) which on average is 0.5 ms. If we read the signal with a limited time resolution, each spike can be assimilated to a Dirac-δ function located at time τ_l and the signal can be represented as

$$S(t)=\Sigma_l\delta(t-\tau_l). \tag{2}$$

The ISI sequence is given by $T_l\{\tau_l-\tau_{l-1}\}$. The information in $S(t)$ is encoded in the moments of all order of the distribution T_l. As was shown elsewhere[10] a driver around the average frequency $1/\langle\text{ISI}\rangle$ will easily synchronize the spike train. Since the "Arnold tongue" corresponding to a perturbation around 1% has a width of 0.5 kHz, any signal within this frequency width can be encoded into the spike train. As we introduce the pulsed perturbation, the phase of the laser locks to the phase of the perturbation, in a 1:1 regime. By 1:1 locking we mean that the standard deviation of the interspike interval T_i minus P is small. This period-1 stable orbit is robust against large variations of ϵ and P, as one can see from the numerical results shown in Fig. 1(a). Note that this large periodic window is still present even in the case where noise is present $(D\neq0)$.[11] The periodic attractor for parameter values inside this period-1 window does not belong to the original invariant dynamics, which means that the perturbation stabilizes a transient dynamics. From now on, we work with the return map of the first return time, that is T_i vs T_{i+1}, where T_i is the time between two successive spikes measured when the laser intensity x_1 overcomes the threshold value $x_1=0.003$. This map, in addition to being topologically equivalent to other threshold maps (like the map of the maxima values of the laser intensity), can be easily obtained from experiment (Ref. 12).

Further, we give a series of definitions in case we want to perturb the laser for values of P within the interval $I=[P^{\min},P^{\max}]$. We divide this interval into N equally spaced subintervals $\{I_0,I_1,\ldots,I_{N-1}\}$ of size I/N, and choose the middle points of each subinterval as the values P^j that P can assume, with $j=0,\ldots,N-1$. So, the P^j are the elements of the discrete set Q. We define a two-dimensional (2D) discrete space composed by N^2 points with coordinates $\{P^m,P^n\}$, with $m,n=0,\ldots,N-1$. For the sake of simplicity we call the point $\{P^m,P^n\}$ as the point mn. We call this space the *input space* and it can be seen in Fig. 1(b). By choosing $I=[110,115]$ [any other interval inside the periodic region of Fig. 1(a) could also be used] and $N=2$ we have the set P formed by the elements $P^0=111.25$ and $P^1=113.75$. We also define the set O formed with the elements P_t, where

$t=1,\ldots,l$, and l is the total number of pulsed perturbations applied to the laser system. Thus, each element in the set O can assume any of the values in the set Q. We encode the set O into a symbolic sequence $X=\{X_1,X_2,\ldots,X_l\}$, composed of integer elements, such that if $P_t=P^m$, then $X_t=m$. In this way, assuming $N=2$, the sequence of perturbations $O=P^0\ P^0\ P^1$ is encoded into the sequence $X=0\ 0\ 1$. This sequence can be represented in the transmitted space by a transition from the point 00 to the point 01. In longer sequences the trajectory in the transmitted space is governed by a shift of one element in this sequence. The allowed transitions are shown in Fig. 1(b).

We verify that changing the parameter P in order to have a discrete random walk in the input space, a complex structure appears in the return map, $T_i\times T_{i+1}$, such as the one shown in Fig. 1(c). In this figure, we see eight subpartitions. This return map, which we call *output map*, is a 2D embedding of the interspike intervals T_n. Analogous to what was done in the input space, we define a 2D trajectory, $R=\{R_1,R_2,\ldots,R_l\}=\{\{T_1,T_2\},\{T_2,T_3\},\ldots,\{T_{l-1},T_l\}\}$. A trajectory R' departing from one of these subpartitions eventually comes back to itself, therefore, this is a recurrent trajectory. This space is not discrete like the input map, and in addition it does not have 2^N points, but 2^{N+1} subpartitions. We also encode this space by calling each one of these subpartitions by a tern of integer numbers mno, with $m,n,o=0,\ldots,N-1$. Assuming we are in the subpartition mno, that means that the laser was perturbed by the sequence of perturbations $P^m\ P^n\ P^o$. So, each subpartition is named such that a point of the trajectory R is iterated in the output map as a shift in the symbolic sequence. For example, if we perturb the system with the symbolic sequence 0100, the laser produces in the output map a trajectory R going from the subpartition 010 to the subpartition 100. This way, for a given received trajectory, we decode the information by constructing its symbolic sequence, $Y=Y_i,Y_{i+1},Y_{i+2},\ldots$, through the following rule. We check in which one of the eight subpartitions mno the point $R_{i+1}=\{T_{i+1},T_{i+2}\}$ is. Then, we assume $Y_i=m$, $Y_{i+1}=n$, and $Y_{i+2}=o$. Next, we check if the point $R_{i+2}=\{T_{i+2},T_{i+3}\}$ falls in $n\ o$ 0 or $n\ o$ 1. In the latter case, we assume $Y_{i+3}=1$, and in the former case $Y_{i+3}=0$. Finally, we define the point T_i^{mno} as the average value of all the T_i points present in the subpartition mno, and the point T_{i+1}^{mno} as the average value of all the T_{i+1} points present in the subpartition mno.

In case one wants to use this laser to transmit information, basically one has to define the perturbations such that a discrete message Y, composed by a discrete alphabet of N elements, is received, with $Y=X$, where X is the message to be transmitted, that we assume to be a random discrete source. We call this procedure the *dynamical encoder*.[3] We define the *dynamical decoder* as the procedure of defining subpartitions to symbolize the received trajectory, allowing one to recover the message X. A trivial dynamical encoder is one that applies a perturbation P^i, if "i" (with $i=1,\ldots,N$) is to be transmitted. Similarly, a trivial decoder is the one that associates the subpartitions mno with the partition no, without considering the structure of the output map. One could wonder if there is any advantage to using this chaotic laser instead of using pulsed signals with different periods to encode information. Suppose we use N different pulsed signals. Each pulse signal contains *at most* $\log_2 N$ bits of information. On the other hand, these recurrent trajectories contain per spike *at least* $\log_2 N$ bits, as we further demonstrate for a binary alphabet, that is $N=2$.

An intelligent dynamical en(de)coder is based on the fact that messages of 6 bits have a unique orbit R with length 4 (an orbit that connects 4 points in the output space). Therefore, knowing R_i and R_{i+4} all the received orbit is known. However, in cases where some information is missing, i.e., when we do not have full knowledge of $R_{i+4}=\{T_{i+4},T_{i+5}\}$, but we know the value of T_{i+5}, we can extract information from the output space. In practice, the return map allows us to recover the symbol that T_{i+4} encodes, and therefore, we obtain the point R_{i+4}. For the sake of simplicity, instead of working with trajectories R of size 4, we work with trajectories R of size 2. Let us assume the desired message to be transmitted is $X=$"1 0 1 0" but the transmitter only sends the first and the last bit. This 4 bit message is the encoding of five interspike intervals. Therefore, the transmitter sends only the first two spikes and the last two spikes. This way, the receiver knows that the symbolic message has to be one of the following sequences: "1 1 0 0," "1 1 1 0," "1 0 1 0," or "1 0 0 0." Although we do not know the interspike intervals T_2 and T_3, which code the second and third bit, we know the time $A=T_2+T_3$. This information can be compared with the sum $B=T_n^{mn0}+T_{n+1}^{mn0}$ (the last bit zero is a known information). We choose the subpartition $mn0$ such that B is the closest to A. This way we define what sequence has been transmitted. Let us assume we have a long message, i.e., $l\geqslant 6$. In that case, every 4 bits, 2 bits do not need to be transmitted, or in terms of spikes, for every three spikes, one spike does not need to be transmitted.

This property can be advantageously used to design a dynamical encoder that improves the rate of transmission per spike or that recovers lost information due to dropouts. In the above-given example, if the receiver can recover all the non-transmitted interspike intervals, the amount of information is $\frac{3}{2}H$ bits per transmitted spike, where $H=1$ is the amount of information of the source (uniform binary source). We can also implement time multiplexing. Within the time interval between the prior and posterior spikes to the one that is not transmitted, a pair of spikes that encodes some other information can be sent. The interspike interval between this pair of spikes should be smaller than $P^{\min}$.

The efficiency of this en(de)coder is measured by the mutual information, H, between the message X and the decoded message Y, which determines the amount of information that the receiver gets from X, knowing Y. Basically, it gives the amount of information after the application of the dynamical decoder,

$$H=\sum_i \sum_j p(X_i,Y_j)\log_2\left(\frac{p(X_i,Y_j)}{p(X_i)p(Y_j)}\right), \tag{3}$$

where $p(X_i,Y_j)$ is the probability of sending X_i and receiving Y_j (transition probability), $p(X_i)$ is the probability of sending X_i, and $p(Y_i)$ is the probability of receiving Y_i.

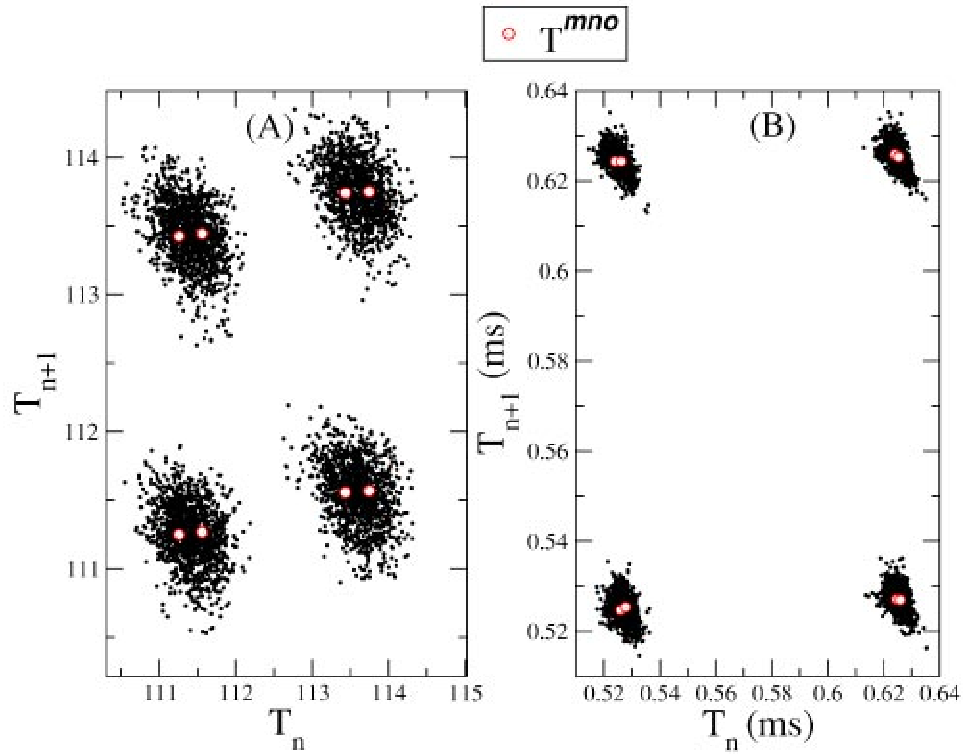

FIG. 2. (Color online) (a) Output map for the simulation of Eq. (2) done with noise amplitude of $D=0.002$, $N=2$, and P values equal to the ones of Fig. 1(b). (b) The received map for $N=2$ from experimental data. In the experiment, $P^0=0.5263$ ms, $P^1=0.6250$.

Figure 1(c) shows an ideal situation. In practice, the amount of internal noise present in the system is much larger, sufficient to merge the subpartitions as shown in Fig. 2(a) for a numerical simulation with $D=0.002$ and the experimental data (b), where we also plot the points T_n^{mno}, T_{n+1}^{mno} that are used to en(de)code the information in (of) the spikes. Assuming we do not transmit one spike out of five, for the data of Fig. 2(a) we can have 87.38% of correction, which means that the amount of information is 1.166 813 4 bits per transmitted spike.

For experimental data we get $H=1.1416$ bits per transmitted spike, also larger than 1, which means that the elements in the alphabet of Y contain more information than the elements in the alphabet of X. Indeed, the mere sequence $S(t)$ of spikes on the X signal must be complemented by the information on the dynamical features of the signal carrier,

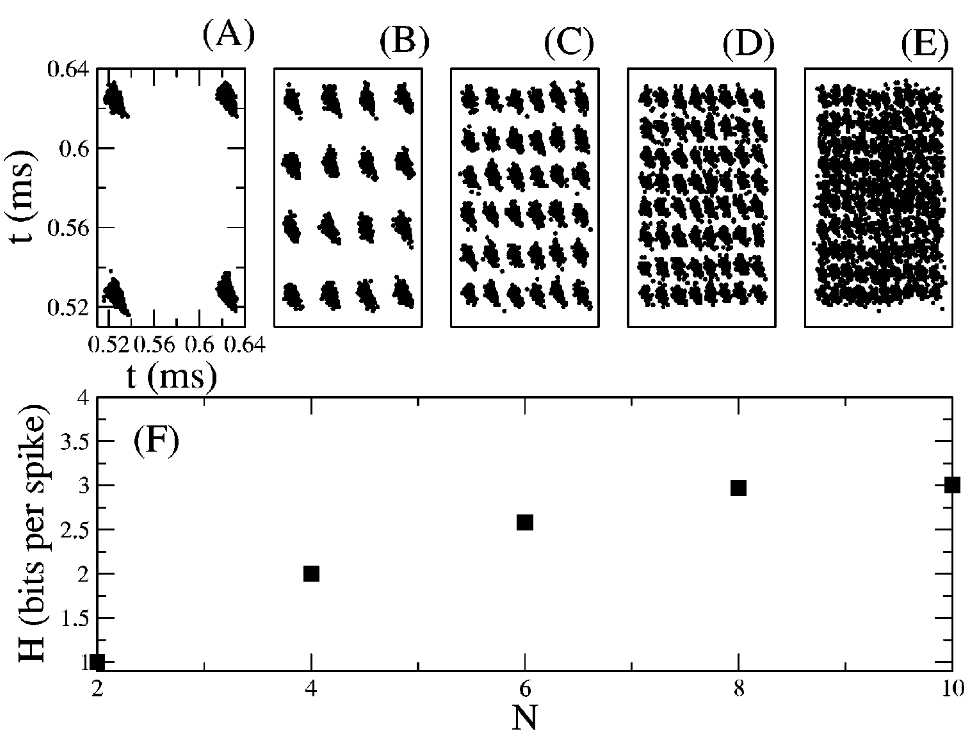

FIG. 3. Experimental output map for $N=2$ (a), $N=4$ (b), $N=6$ (c), $N=8$ (d), and $N=10$ (e). The mutual information between the transmitted symbolic sequence and the received symbolic sequence, or the amount of information received per transmitted spike with respect to N is shown in (f).

which can be exploited as explained here in the following. In case one wants to transmit a message composed by an alphabet that contains more than g elements, with $g>2$, we can use $N=g$ different perturbations selected accordingly to create at least g discernible partitions in the output map. For simplicity we consider only the partitions in the received space named with two symbols mn, with $m,n=1,2,\ldots,g$, that is we ignore the existence of g subpartitions for each partition mn. In Fig. 3(a) we show output maps from experiments for $N=2,4,6,8,10$, and in (b) we show that the mutual information between X and Y with respect to N is $\log_2(N)$, up to $N=8$. For $N=10$, $H<\log_2(10)$ due to the fact that the partitions collide with each other. However, one would still prefer to use $N=10$, instead of $N=8$, once $H(N=10)>H(N=8)$. That is due to the fact that the amount of information gained per using two more partitions is bigger than the loss of information due to the partition border collision. In fact, this loss of information is small because most of the points in the output map are far from the boundaries.

Concluding, we have implemented (numerically and experimentally) a simple technique for controlling a homoclinic chaotic laser, which enables one to construct a communication system such that the carrier signal (the laser spiking signal) contains more information than the source it encodes. This extra information can be used to recover lost information, or to increase the ratio of information per transmitted spike. This is only possible because the controlled trajectory still preserves important dynamical characteristics like smoothness, recurrence, and short term memory. Finally, in this work we have assumed that only one subpartition exists. But in fact, there is a hierarchical fractal-like structure of subpartitions within each partition. Clearly, these complex structures could be explored in order to increase even more the efficiency of this proposed communication.

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