

In phase and anti-phase synchronization of coupled homoclinic chaotic oscillators

I. Leyva^{1,3,*}, E. Allaria¹, S. Boccaletti¹, and F.T. Arecchi^{1,2}

¹ *Istituto Nazionale di Ottica Applicata, Largo E. Fermi 6, 50125 Florence, Italy*

² *Department of Physics, University Of Firenze, Italy and*

³ *Universidad Rey Juan Carlos. c/Tulipan s/n. 28933 Mostoles Madrid, Spain*

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We numerically investigate the dynamics of a closed chain of unidirectionally coupled oscillators in a regime of homoclinic chaos. The emerging synchronization regimes show analogies with the experimental behavior of a single chaotic laser subjected to a delayed feedback

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The study of collective phenomena in chains of chaotic oscillators has recently attracted a wide interest, for the variety of possible scenarios that can be found, and the analogies with biological complex systems. Experiments have been performed on arrays of optical systems [1], electronic circuits [2], neurons [3, 4] and chemical oscillators [5], reporting different synchronization patterns, such as antiphase synchronization, and clustering [6]. In a theoretical work, the dynamics of an unidirectionally coupled ring of chaotic systems has been explored in view of possible applications to neural systems [7]. However, the need for parameter accurate control has up to now limited experiments to a small number of oscillators, hence most works on large chains are only numerical. In addition, an analogy between spatially extended systems and delayed systems has already been drawn [9].

The aim of this paper is to show the equivalence between the dynamics of one delayed system and a unidirectionally coupled ring of identical systems. A suitable framework for this symmetry makes it possible to retrieve the dynamics of an arbitrary large array of coupled systems by means of a single delayed system.

Our starting point is the recent report of the Delayed Self Synchronization (DSS) of chaotic homoclinic spike trains in a CO₂ laser [12]. It has been shown that this system displays a continuous return to a saddle focus where it has a large sensitivity to external perturbations [13].

For convenience we report a stretch of the experimental time signal of the homoclinic intensity (Fig. 1 (a)) as well as its 2D phase space projection. (Fig. 1 (b)) obtained by an embedding technique and representing a superposition of a long sequence of spikes. We realize that on each cycle the intensity returns to its zero value (baseline of Fig 1 (a) and point O in Fig 1 (b)) emerging with a large spike followed by a decaying spiral toward a saddle region S. The escape from S is represented by a growing oscillation, which appears as a nesty region in Fig 1 (b). Notice that the saddle region S is very contracted in phase space but stretched in time-wise, viceversa the spike P occurs over a long time but is spread over a large space region.

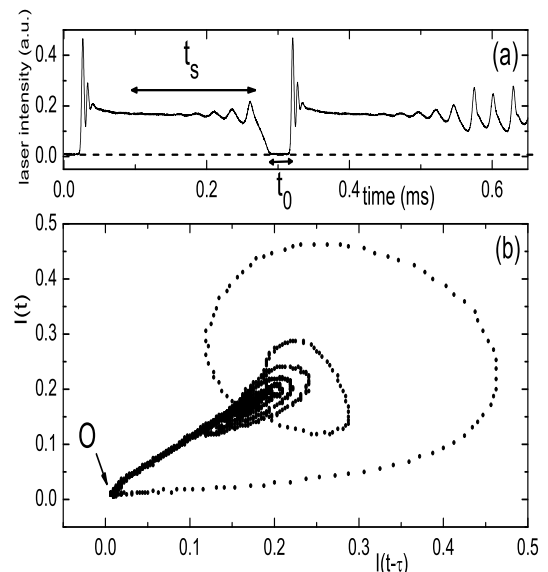


FIG. 1: (a) Experimental time intensity in the homoclinic regime. The dashed line corresponds to the baseline. (b) Phase space projection of the experimental time series.

The chaotic characteristic of the inter-spikes interval (ISI) is due to the chaotic permanence time t_s around the saddle point S [10], whereas the permanence time t_o on the baseline is fixed. This is confirmed by the values of the local Lyapunov exponents in the two regions [11].

In this condition, a long-delayed feedback signal can stabilize complex periodic orbits of period T characterized by a pseudo-chaotic train of pulses, that is, by a limited sequence with chaotic ISI, that repeats after a time T again for ever.

This ability of DSS can be of interest in relation to recent studies on neuronal transformation mechanisms of short-term memories into permanent (long term) memories via the so called synaptic reentry reinforcement [14]. Therefore, the possibility to describe DSS as a synchro-

*Electronic address: ileyva@ino.it

nized state in a closed chain of oscillators is a relevant issue.

First it is important to establish under what conditions a chain of coupled oscillators can be equivalent to a delayed system. Since the delay implies that the information propagates in one direction, just unidirectional coupling will be considered in the oscillators chain, so that the oscillator at the site i is driven by the previous one at the site $i - 1$. In addition, the delayed reentry of the signal in the system means that the system is exposed to the total information generated over a previous time stretch of size T_d . Meeting this condition imposes a closed boundary with the last oscillator coupled to the first one.

With these boundary and coupling constraints, we build the array by using the scaled equations that model the experimental laser system [15]

$$\begin{aligned}
 \dot{x}_1^i &= k_0 x_1^i (x_2^i - 1 - k_1 \sin^2 x_6^i), \\
 \dot{x}_2^i &= -\gamma_1 x_2^i - 2k_0 x_1^i x_2^i + g x_3^i + x_4^i + p, \\
 \dot{x}_3^i &= -\gamma_1 x_3^i + g x_2^i + x_5^i + p, \\
 \dot{x}_4^i &= -\gamma_2 x_4^i + z x_2^i + g x_5^i + z p, \\
 \dot{x}_5^i &= -\gamma_2 x_5^i + z x_3^i + g x_4^i + z p, \\
 \dot{x}_6^i &= -\beta (x_6^i - b_0 + \frac{r(x_1^i - \epsilon x_1^{i-1})}{1 + \alpha x_1^i}),
 \end{aligned} \tag{1}$$

where the index i denotes the i^{th} site position, and for each oscillator x_1 represents the laser intensity, x_2 the population inversion between the two resonant levels, x_6 the feedback voltage which controls the cavity losses, while x_3, x_4 and x_5 account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band.

In analogy with the experiment, the coupling on each oscillator has been realized by adding a function of the intensity (x_1) of the previous oscillator to the equation of its feedback signal x_6 .

The parameters are the same for all elements of the chain. Here, k_0 is the unperturbed cavity loss parameter, k_1 determines the modulation strength, γ_1, γ_2, g are relaxation rates, p_0 is the pump parameter, z accounts for the number of rotational levels, β, b_0, r, α are respectively the bandwidth, the bias voltage, the amplification and the saturation factors of the feedback loop, and ϵ is the coupling strength. The values used in the numerical simulation to reproduce the regime of homoclinic chaos [15], are: $k_0 = 28.5714, k_1 = 4.5556, \gamma_1 = 10.0643, \gamma_2 = 1.0643, g = 0.05, p_0 = 0.016, z = 10, \beta = 0.4286, \alpha = 32.8767, r = 160, b_0 = 0.1032$.

The coupling strength ϵ is the control variable and it can assume both negative and positive values, as in the experiment. An important feature found in the experiment is that the route towards the pseudo-chaotic state depends on the sign of the delayed feedback modulation. If positive, the signal tends to reach in-phase synchronization with the delayed modulation, while if it is negative, the coupling comes out to be phase-repulsive and

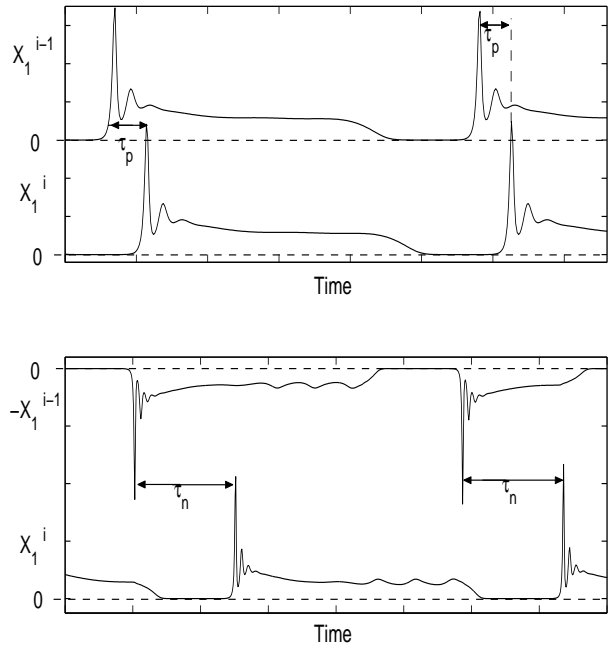


FIG. 2: Detail of the time profile of the i^{th} (solid line) and $(i - 1)^{th}$ elements of the chain, for (a) positive coupling and (b) negative coupling.

the signal is in anti-phase with the feedback perturbation.

This route is not symmetric for the coupling strength: a significantly stronger coupling is needed to reach the pseudo-chaotic state for a positive than for a negative feedback signal. This behavior is a consequence of the fact that the system is more efficiently removed from the saddle focus neighborhood for a decrease of the modulating signal. Eventually, when the coupling is strong enough, the system becomes fully periodic for both positive and negative values, i.e. there is no longer a pseudo-chaotic status.

The experimental data [12] reveal a small time offset between the modulation and the signal, which is independent of the long delay T_d and of the coupling strength. This offset depends on the sign of the modulation feedback, and it has been measured to be $\tau_n = 140 \mu s$ for negative coupling and $\tau_p = 20 \mu s$ for a positive, against an average ISI of $500 \mu s$.

The numerical results explain how these asymmetries depend on the coupling sign. In both cases, the slave intensity lags with respect to the driving one, but by different amounts. Precisely, the driver's negative slope forces the slave to precipitate from the saddle region to the zero baseline. In case (a), the driver's and slave's collapses onto the zero baseline occur just one after the other and the negative oscillations around the spike do not induce a transition insofar as they occur while the slave is already on the baseline. On the contrary, in case

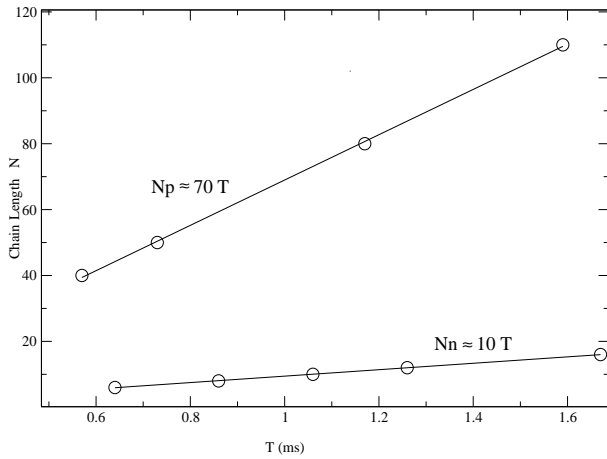


FIG. 3: Different response of the chain for negative (N_n , circles) and positive (N_p , squares) coupling. The overall propagation velocities N_j/T ($j=n,p$) are in the ratio 1/7 as the two offsets time in Fig. 2

(b) the first negative driver's slope which can force the slave to fall away from the saddle coincides with the first negative spike oscillation. As a result, considering that the permanence time on the baseline t_o is a fixed fraction of the whole period, two different offsets $\tau_p < \tau_n$ occur.

Such an interpretation based on the numerical results can be also applied to the case of the DSS where the delayed laser intensity stays for the driver signal.

Therefore we can say that the times τ_p and τ_n correspond to different information propagation velocities along the array, with a lower velocity for the negative coupling.

In order to model the laboratory system with an array of N oscillators, we must match the overall delay along the closed chain $N\tau_j$ (where j stays for p or n depending on the coupling) with a delayed time T_d so that each system is exposed to a delayed version of its own signal $x_1(t - T_d) = x_1(t - N\tau_j)$. As $\tau_p < \tau_n$, a longer chain will be needed in a positive case to obtain the same periodicity. The different propagation velocities can be seen in Fig.3 where the relation between the repetition time T and the oscillators number of a chain N are reported for the two conditions. From Fig. 3 we can also establish that $\tau_n \simeq 7\tau_p$, which fits well with the experimental ratio 140/20. An example of the dynamical characteristics can be better observed in the numerical case of Fig. 4. Here, the time intensity profile of a single site of the chain is compared with its driver neighbor in the pseudochaotic regime, for both positive (Fig. 4(a)) and negative (Fig. 4(b)) coupling. The chains have been chosen to have approximately the average period $\langle T \rangle = 0.95$ ms, and $\langle T \rangle = 1.25$ respectively, with $N = 7$ in the negative case and $N = 70$ in the positive.

We can already establish a full equivalence between the experimental laser system of Ref. [12] and the closed chain. In Fig. 5 a chain of $N = 8$ coupled negatively

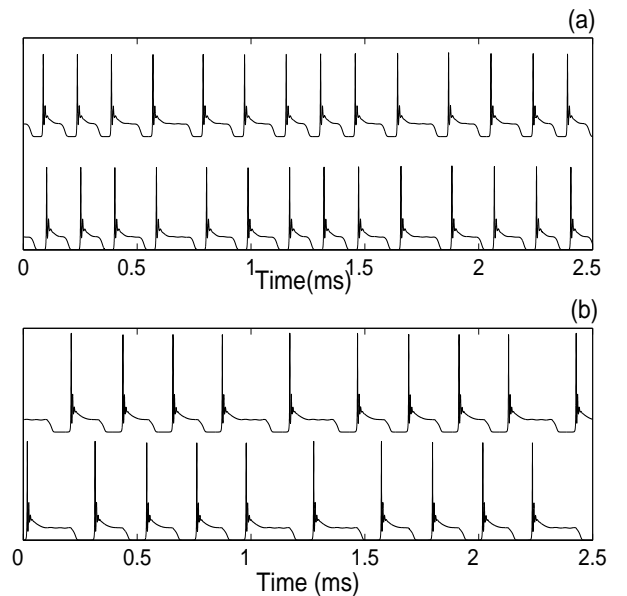


FIG. 4: Intensity profile of two neighbor oscillators (a) $N = 70$, $\epsilon=0.15$, (b) $N = 7$, $\epsilon=-0.042$. The data have been vertically shifted for a better display.

elements is compared to an experiment performed with negative feedback and a delay time of 4 ms, and another chain of $N = 24$ elements with an experiment in which a delay of 12 ms was used. For the experimental case we show the Space Time Representation of the laser intensity [9] (Fig.5(a) and (c)), and for the numerical case we report in gray scale the intensities for the oscillators as a function of time (Fig. 5 (b) and (d)). The experimental results correspond to a negative coupling of -7% , while the numerical results are obtained for $\epsilon = -0.046$, these values being close to the threshold values to establish a stable pattern in the dynamics.

This same behavior in which a positive coupling can induce a collective synchronous behavior, while the negative coupling induces an antiphase dynamics has been observed in experiments in arrays of neurons where the coupling could be modified [16].

In this work we demonstrated that in certain conditions a time delayed system can be used for an experimental verification of the synchronization behavior of arrays of chaotic oscillators. In an example of this equivalence for a delayed laser in the homoclinic chaotic regime, phase and antiphase pseudo-chaotics synchronization regimes have been reported.

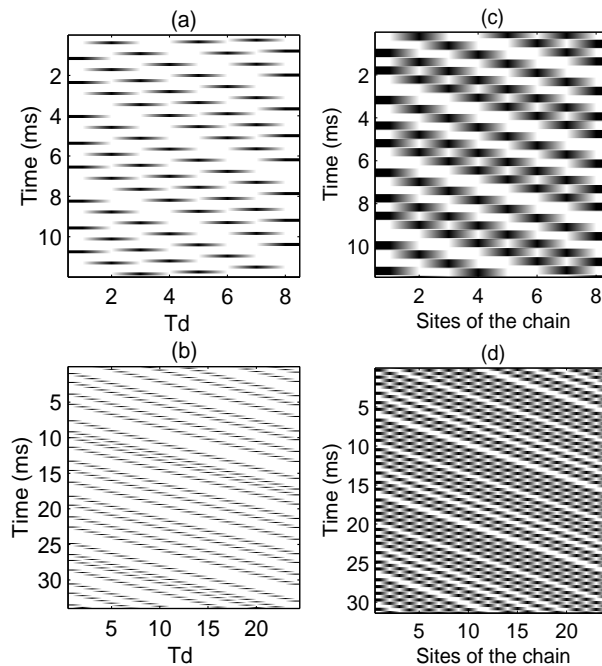


FIG. 5: Space time representation of the experimental laser intensity in the experiments of Ref. [12] (a,c) compared with the numeric result of a oscillators chain (b,d). (a) $T_d=4$ ms, coupling strength -7% (c) $T_d=12$ ms, coupling strength -7%. For the equivalent chain of oscillators: (b) $N=8$, $\epsilon = -0.042$ (d) $N=24$, $\epsilon = -0.042$

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