Atomic density fluctuations in Bose-Einstein condensates

A. Montina and F. T. Arecchi
Dipartimento di Fisica, Università di Firenze, Via Sansone 1, 50019 Sesto Fiorentino (Firenze), Italy
(Received 18 June 2003; published 17 November 2003)

A Yukawa-like equation, introduced to evaluate the first order correlation function in a Bose-Einstein condensate, can be used to evaluate the atomic density fluctuations. It does not require a few mode approximation. We apply this approach to calculate the atomic fluctuations of a condensate of attractive atoms in a double-well potential. The atomic system undergoes a spatial symmetry breaking above a critical number N_c of atoms. Near N_c the atomic fluctuation δN inside a single well becomes very large and experimentally observable by a phase-contrast imaging technique. δN is related to the collective mode temperature, which may be different from the condensate temperature measured by means of the thermal tails. A one-dimensional array of wells is also considered.

DOI: 10.1103/PhysRevA.68.053608 PACS number(s): 03.75.Hh, 03.75.Lm

I. INTRODUCTION

An ensemble of Bose atoms is described by quantum field operators. In a Bose-Einstein condensate (BEC) the thermal fluctuations of this field are reduced so much that a long-range order appears. Many properties of this degenerate gas are derivable by the mean field Gross-Pitaevskii equation, which neglects the quantum and thermal fluctuations. The long-range spatial order has been studied in a series of theoretical papers [1–7]. On the experimental side, interference experiments involving sodium [8] and rubidium [9] condensates have demonstrated the presence of long-range order. Subsequent experiments have explored some features of second order [10] and third order [11] atomic coherences. In Ref. [10] the relationship between the second order coherence and the interaction energy was studied, inferring that measurements of the released energy are consistent with a unit value for the second order coherence of a pure condensate. Burt *et al.* [11] measured the three-body rubidium recombination rate of a cold noncondensate and of a condensate [11], finding that the corresponding ratio of the third order coherences is 7.4 ± 2.6, in agreement with the predicted value of 6. All these experiments show that the field fluctuations are negligible. Typically, the fluctuations of density and phase are important near the BEC transition temperature. By contrast, large thermal phase fluctuations are predicted in very elongated trapped condensates [12]. These fluctuations have been observed experimentally in a ^{87}Rb BEC [13]. Phase fluctuations have also been observed in a one-dimensional array of weakly linked traps [14]. They are due to the atom-atom repulsive interaction. In Ref. [15] the transition from a superfluid to a Mott insulator in a three-dimensional lattice potential has been observed. This occurs when the atomic fluctuations in a single well are suppressed and the relative phase is completely indeterminate. The problem of the relative phase of a condensate of repulsive atoms in a double-well potential is studied, for example, in Ref. [16].

In this work, we show that a Yukawa-like equation, introduced in Refs. [17,18] to calculate the first order correlation function, can be used to evaluate the density fluctuations. As an application, we consider a condensate of ^{7}Li in a double-well potential, which undergoes a spatial symmetry breaking of the atomic density for a suitable number N_c of atoms [19]. A symmetry breaking can occur also in the internal states of a two-component condensate [20]. We evaluate the fluctuation δN of the number of atoms in a single well. Near N_c, δN can be very large. It is experimentally observable by a phase-contrast imaging technique. We extend our calculations to a one-dimensional array of traps. δN is related to the collective mode temperature, which may be different from the condensate temperature measured by means of the thermal tails, since the collective mode may be weakly coupled with the other ones. For this reason, an experimental measure of δN would be interesting. A weak coupling is important to reduce the decoherence effects in the macroscopic quantum superposition experiments proposed in Refs. [19,21].

Note that the Yukawa-like equation of Refs. [17,18] is obtained by using the Bogoliubov approximation [22]. This approximation is suitable for very low temperatures. However, as pointed out in Ref. [23], due to the presence of a trap, the condensed particles form a high-density cloud with a low-density background of noncondensed particles. So even with a significant fraction of noncondensed atoms the Bogoliubov approximation can work. The numerical calculation of Ref. [24] confirms this expectation.

The Yukawa-like equation is useful when, with the standard approach, we have to diagonalize a large matrix, as occurs in a tridimensional problem with a fine structure potential (for example, in a linear array of tridimensional potential wells or in a lattice).

In Sec. II we establish the equations for atomic fluctuations; in Sec. III we show two methods to solve them. In Sec. IV we report the results obtained for a condensate of ^{7}Li atoms in a double-well potential. In the last section the double well is replaced by an array.

II. DENSITY FLUCTUATIONS

A Bose-Einstein condensate is described by a quantum field $\hat{\psi}(\vec{x})$ that satisfies the well-known bosonic commutation rules. This field can be written as

$$\hat{\psi}(\vec{x}) = \psi(\vec{x}) + \hat{\phi}(\vec{x}), \tag{1}$$

where $\hat{\phi}(\vec{x})$ is a field operator, which satisfies the same commutation rules as $\hat{\psi}(\vec{x})$, and $\psi(\vec{x})$ is a *c*-number field. We set ψ as the solution of the time-independent Gross-Pitaevskii equation [26], i.e.,

$$\left[-\frac{\hbar^2}{2m}\nabla^2+V(\vec{x})+g|\psi|^2-\mu\right]\psi\equiv H_{GP}\psi=0, \tag{2}$$

μ being the chemical potential of the atomic gas and $g=4\pi\hbar^2 a/m$, where a is the *s*-wave scattering length. In this case and for sufficiently small temperatures, the operator $\hat{\phi}$ can be regarded as a perturbation, thus the Hamiltonian is approximated by a quadratic operator in $\hat{\phi},\hat{\phi}^\dagger$ (Bogoliubov approximation) [22]. In a previous paper [17] one of us introduced a suitable method to evaluate the expectation values

$$C_1(\vec{x},\vec{y})=\langle\hat{\phi}^\dagger(\vec{x})\hat{\phi}(\vec{y})\rangle, \tag{3}$$

$$C_2(\vec{x},\vec{y})=\langle\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})\rangle. \tag{4}$$

For this purpose we used both the standard symmetry breaking Bogoliubov approximation and the number conserving approach of Ref. [25]. In this article we refer to this latter case.

The density fluctuations can be evaluated from $C_{1,2}$, as follows. Experimentally the atomic density is measured with a finite spatial resolution; thus the physical quantity of interest is

$$\hat{n}_m\equiv\int\Omega(\vec{x})\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})d^3x, \tag{5}$$

where Ω is a peaked normalized function, whose width σ_Ω establishes the spatial resolution.

The density fluctuation is

$$\begin{aligned}(\delta n_m)^2\equiv\langle\hat{n}_m^2\rangle-\langle\hat{n}_m\rangle^2=&\int d^3x\,d^3y\,\Omega(\vec{x})\Omega(\vec{y})\\&\times[\langle\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})\hat{\psi}^\dagger(\vec{y})\hat{\psi}(\vec{y})\rangle-\langle\hat{\psi}^\dagger(\vec{x})\hat{\psi}(\vec{x})\rangle\\&\times\langle\hat{\psi}^\dagger(\vec{y})\hat{\psi}(\vec{y})\rangle].\end{aligned} \tag{6}$$

We can use Eq. (1) to write $(\delta n_m)^2$ in terms of ψ, ψ^*, $\hat{\phi}$, and $\hat{\phi}^\dagger$. If the fluctuations are small with respect to the mean field then we can neglect the quantity $\langle\hat{\phi}^\dagger(\vec{x})\hat{\phi}(\vec{x})\hat{\phi}^\dagger(\vec{y})\hat{\phi}(\vec{y})\rangle$; thus we have that

$$(\delta n_m)^2=\int d^3x\,d^3y\,\Omega(\vec{x})\Omega(\vec{y})d(\vec{x},\vec{y}), \tag{7}$$

where

$$\begin{aligned}d(\vec{x},\vec{y})\equiv&\psi(\vec{x})\psi^*(\vec{y})\langle\hat{\phi}^\dagger(\vec{x})\hat{\phi}(\vec{y})\rangle+\psi^*(\vec{x})\psi^*(\vec{y})\\&\times\langle\hat{\phi}(\vec{x})\hat{\phi}(\vec{y})\rangle+\text{H.c.}+|\psi(\vec{x})|^2\delta(\vec{x}-\vec{y}).\end{aligned} \tag{8}$$

For the minimal energy state of a trapped system, we can consider a real ψ without loss in generality. In the following we refer to this particular case, but the generalization is trivial. Equations (7) and (8) then become

$$(\delta n_m)^2=4\int d^3x\,d^3y\,\Omega(\vec{x})\Omega(\vec{y})\psi(\vec{x})\psi(\vec{y})C_R(\vec{x},\vec{y}), \tag{9}$$

where $C_R(\vec{x},\vec{y})\equiv\langle\hat{\phi}_R(\vec{x})\hat{\phi}_R(\vec{y})\rangle$, $\hat{\phi}_R$ being given by $(\hat{\phi}+\hat{\phi}^\dagger)/2$. Let us define the following operators:

$$H_{1,2}\equiv\mathcal{L}^s_\epsilon\pm g\psi^2, \tag{10}$$

$$A_T\equiv\tanh[(\beta/2)(H_1H_2)^{1/2}], \tag{11}$$

where $1/\beta$ is the condensate temperature in energy units and

$$\mathcal{L}^s_\epsilon\equiv-\frac{\hbar^2}{2m}\nabla^2+2g|\psi|^2+V-\mu+\epsilon, \tag{12}$$

ϵ being an infinitesimal positive number introduced in Ref. [17] to eliminate some divergences arising in the intermediate calculations. In the case considered here, we can set $\epsilon=0$ directly. If F^S is the solution of the operator equation

$$H_1F^S=\frac{1}{4}A_T^{-1}(H_1H_2)^{1/2} \tag{13}$$

and

$$Q\equiv\mathbb{1}-|\psi\rangle\langle\psi|,$$

then we have [17]

$$(QF^SQ)(\vec{x},\vec{y})=\langle\hat{\phi}_R(\vec{x})\hat{\phi}_R(\vec{y})\rangle\equiv C_R(\vec{x},\vec{y}). \tag{14}$$

In Ref. [17] we showed that if we integrate over one of the tridimensional variables $\vec{x},\vec{y}$ of $F^S(\vec{x},\vec{y})$ with a smoothing weight function, as was done in Eqs. (9) and (14), then F^S can be evaluated, for sufficiently high temperatures, using the expansion (the validity condition is also discussed at the end of Sec. IV)

$$A_T^{-1}(H_1H_2)^{1/2}\simeq 2kT+\frac{1}{6}\beta H_1H_2+\cdots. \tag{15}$$

Retaining only the first term, Eq. (13) becomes

$$\begin{aligned}H_1F^S=\frac{1}{2}kT\Rightarrow&\left[-\frac{\hbar^2}{2m}\nabla^2_{\vec{x}}+V(\vec{x})+3g\psi^2(\vec{x})-\mu\right]F^S(\vec{x},\vec{y})\\&=\frac{1}{2}kT\delta(\vec{x}-\vec{y}).\end{aligned} \tag{16}$$

Thus, the main problem is to solve Eq. (16). Once we have evaluated F^S, we have to calculate C_R using Eq. (14). Finally, from C_R we can find $(\delta n_m)^2$ using Eq. (9). In the next section we provide two approaches to solving Eq. (16); the first one has already been proposed in Ref. [17].

As said in the Introduction, Eq. (16) relies on the Bogoliubov approximation, which is suitable for very low temperatures. However, as was pointed out in Ref. [23], due to

the presence of a trap, the condensed particles form a high-density cloud with a low-density background of noncondensed particles. This low-density background has a small influence on the condensate low-frequency modes, which give the main contribution to the fluctuation of the integrated atomic density of Eq. (5) [see Eq. (7)]. So even with a significant fraction of noncondensed atoms the Bogoliubov approximation can work, as confirmed by the numerical calculation of Ref. [24].

In Secs. IV and V, we will apply our equation to the case of a condensate of attractive atoms. This system undergoes a symmetry breaking for a suitable number of atoms. Near the transition, a large collective fluctuation δN of the density in the single well is present. If it is too large, the cubic and quartic terms neglected in the Hamiltonian can become important and the Bogoliubov approximation is no longer valid. To apply the Bogoliubov approximation, the condition that has to be satisfied is $\delta N \ll N$, N being the total number of atoms. In Sec. IV this last condition will be discussed extensively.

In the Bogoliubov approximation we can evaluate the density fluctuations merely by calculating the eigenvectors of the following matrix:

$$\mathcal{L}=\begin{pmatrix} H_{GP}+g|\psi|^2 & g\psi^2 \\ -g(\psi^*)^2 & -[H_{GP}+g|\psi|^2] \end{pmatrix}. \tag{17}$$

It is the standard approach. However, for a tridimensional problem without particular symmetries, this calculation can be very expensive or impractical. By contrast, the resolution of Eq. (16) requires much less resources.

Denoting by ϵ_k and ϕ_k the eigenvalues and eigenvectors of the operator H_1, from Eq. (16) we can write F^S explicitly,

$$F^S(\vec{x},\vec{y})=\frac{1}{2}kT\sum_k \epsilon_k^{-1}\phi_k(\vec{x})\phi_k(\vec{y}). \tag{18}$$

Thus, in contrast with $\mathcal{L}$, the eigenvalues of H_1 have a direct relation with the density fluctuations. The eigenvalues of $\mathcal{L}$ and H_1 can have different behaviors (For example, see the lines that drop to zero in Fig. 4. This figure is discussed in Sec. V.)

III. SOLUTION METHODS

In the following subsections we describe two methods to solve Eq. (16).

A. Relaxation method

Let us define the function

$$\overline{F^S}(\vec{x})\equiv\langle\vec{x}|F^S Q\int d^3y|\vec{y}\rangle\psi(\vec{y})\Omega(\vec{y}). \tag{19}$$

From Eq. (16) we find that

$$\left[-\frac{\hbar^2}{2m}\nabla_{\vec{x}}^2+V(\vec{x})+3g\psi^2(\vec{x})-\mu\right]\overline{F^S}(\vec{x})=S(\vec{x}), \tag{20}$$

where

$$S(\vec{x})=\frac{1}{2}kT\langle\vec{x}|Q\int d^3y|\vec{y}\rangle\psi(\vec{y})\Omega(\vec{y})$$
$$=\frac{1}{2}kT\left[\Omega(\vec{x})-\int\psi^2(\vec{y})\Omega(\vec{y})d^3y\right]\psi(\vec{x}). \tag{21}$$

We find from Eq. (9) that the density fluctuations are given by

$$(\delta n_m)^2=4\int d^3x\,\Omega(\vec{x})\psi(\vec{x})\overline{F^S}(\vec{x}). \tag{22}$$

We can calculate $\overline{F^S}$ using a relaxation method [17]. In fact, if H_1 has positive eigenvalues, $\overline{F^S}$ is the stationary solution ($\tau\to\infty$) of the time-dependent equation

$$\left[\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla_{\vec{x}}^2+V(\vec{x})+3g\psi^2(\vec{x})-\mu\right]\overline{F^S}(\vec{x})=S(\vec{x}). \tag{23}$$

Equation (23) can be solved numerically on a lattice. The numerical integration of Eq. (23) is typically much more efficient than the evaluation of the eigenvectors of $\mathcal{L}$. The drawback of this method is the necessity for H_1 to have positive eigenvalues. For example, this does not apply to negative scattering lengths (which we consider in Secs. IV and V). For this reason, we describe an alternative approach.

B. Low k spectrum evaluation

The eigenfunctions ϕ_k with many nodes in the spatial interval σ_Ω, where the distribution Ω is located, do not contribute to the density fluctuation of Eq. (9); thus it is sufficient to evaluate only the lower portion of the spectrum. The lowest eigenvectors can be calculated in two ways. ϕ_0 can be obtained from the following evolution:

$$\left[\frac{\partial}{\partial\tau}-\frac{\hbar^2}{2m}\nabla_{\vec{x}}^2+V(\vec{x})+3g\psi^2(\vec{x})-\mu\right]\phi(\vec{x},\tau)=0. \tag{24}$$

We have $\phi_0\propto\lim_{\tau\to\infty}\phi$. The other eigenvectors are obtained recursively, $\phi_1\propto\lim_{\tau\to\infty}e^{\epsilon_1\tau}(\phi-k_0\phi_0)$, $\phi_2\propto\lim_{\tau\to\infty}e^{\epsilon_2\tau}(\phi-k_0\phi_0-k_1\phi_1)$, . . . , where $k_i=\langle\phi_i|\phi\rangle$.

Alternatively, we can use the *implicitly restarted Arnoldi method* (IRAM). A FORTRAN library called ARPACK is freely available and uses it. With ARPACK one can choose the eigenvectors of interest (eigenvalues with smallest/largest value or with smallest/largest magnitude) and their number. Note that in the case of $\mathcal{L}$ also only some eigenvectors contribute to the density fluctuations of Eq. (9); however, in contrast with the operator H_1, the spectrum of $\mathcal{L}$ is symmetrically distributed with respect to zero and the eigenvectors of interest have eigenvalues with the smallest magnitude. Thus, Eq. (24) is not useful to evaluate these eigenvectors. Alternatively, one could use for $\mathcal{L}$ the library ARPACK with the option ''smallest magnitude,'' but it is much more efficient with the option ''smallest value,'' which is used for H_1. For a tridimensional problem without symmetry or with a fine spatial structure,

e.g., an array, the use of ARPACK for the operator $\mathcal{L}$ is impracticable. Furthermore, the eigenvalues of H_1 have a physical interpretation, since they are directly related to the density fluctuations, as pointed out at the end of Sec. II.

In some cases, a hybrid of the eigenvector determination and the relaxation method appears as the best solution. For example, the first one can be used to calculate the eigenvectors with negative eigenvalues, which blow up with the relaxation method. The latter can be used to evaluate the contribution of the remaining eigenvectors. In practice, let $\phi_{k=0,\ldots,\bar{k}}$ be the calculated eigenvectors. After each integration step of Eq. (23), we clean up $\overline{F}^S$ in the following way:

$$\overline{F}^S(\vec{x}) \to \overline{F}^S(\vec{x}) - \sum_{k=0,\ldots,\bar{k}} \phi_k(\vec{x}) \int \phi_k(\vec{y}) \overline{F}^S(\vec{y}) d^3y. \tag{25}$$

IV. DOUBLE-WELL POTENTIAL AND NEGATIVE SCATTERING LENGTH

We consider a condensate with attractive atoms trapped in the potential

$$V(\vec{x}) = \frac{1}{2} m[\omega_\parallel^2 x_1^2 + \omega_\perp^2 (x_2^2 + x_3^2)] + A \cos\left(2\pi \frac{x_1}{\sigma}\right). \tag{26}$$

The quadratic part is due to the interaction of the atoms with the magnetic field of the trap. The additional term can be generated by two opposite laser beams in a standing-wave configuration. A suitable choice of the standing-wave parameters yields a double-well potential. We refer to the case of Ref. [19], i.e., we study a condensate of ^{7}Li, with $\omega_\parallel = 2\pi \times 130\ \mathrm{s}^{-1}$ and $\omega_\perp = 2\pi \times 150\ \mathrm{s}^{-1}$. For the standing wave, we set $A/\hbar = 2000\ \mathrm{s}^{-1}$ and $\sigma = 5\ \mu$m. A spatial symmetry breaking occurs for $N \simeq 640 \equiv N_c$. We want to evaluate the atomic fluctuations in a single well near this critical point.

It is evident that at least an eigenvalue of H_1 is negative. Indeed, $H_1 - H_{GP} = 2g|\psi|^2$, g being negative, and the lowest eigenvalue of H_{GP} is zero. For this reason we cannot use only the relaxation method but need to evaluate, at least, the eigenvectors with a negative eigenvalue. In Fig. 1 we report the first two eigenvalues of H_1 as a function of the number of atoms.

The lowest eigenvalue ϵ_0 is always negative and tends to zero for $N \to 0$, since $H_1 \to H_{GP}$. The second eigenvalue ϵ_1 decreases as long as the number of atoms is below the symmetry breaking critical value. For $N \to N_c \epsilon_1 \to 0$, and thus the thermal fluctuations blow up [see Eq. (18)]. ϵ_1 is approximatively a linear function of N; thus we can write

$$\epsilon_1 \simeq \hbar \omega_c \left[1 - \frac{N}{N_c}\right], \tag{27}$$

where ω_c is the collective oscillation frequency for a low number of atoms. The main contribution, say δN_p, to the atomic fluctuations in a well is due to the eigenvector 1; thus

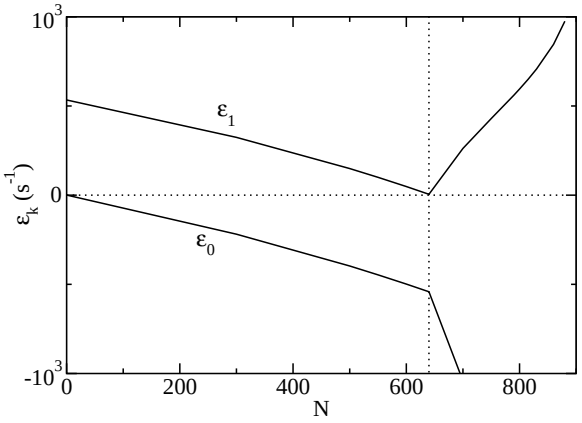

FIG. 1. First two eigenvalues of H_1 as a function of N for a condensate of attractive atoms (^{7}Li) in a double-well potential.

$$\delta N_p = \left(\frac{2kT}{\epsilon_1}\right)^{1/2} \left| \int_V (Q\phi_1)(\vec{x}) \psi(\vec{x}) d^3x \right| \approx \left(\frac{kT}{2\epsilon_1} N\right)^{1/2}, \tag{28}$$

where V is the half space that contains a well (Ω is equal to 1 for $x_1 > 0$, or $x_1 < 0$, and 0 elsewhere). We have conjectured that $|\int_V (Q\phi_1)(\vec{x}) \psi(\vec{x}) d^3x| \approx 0.5\sqrt{N}$, since both ψ and ϕ_1 have a single peak in V and are symmetric and antisymmetric, respectively, in the axial direction. Using Eqs. (27) and (28), we have approximately

$$\delta N_p \approx \left(\frac{kT}{2\hbar\omega_c}\right)^{1/2} \left(\frac{NN_c}{N_c - N}\right)^{1/2}. \tag{29}$$

The parameters N_c and ω_c are not independent. N_c increases on reducing ω_c. For $N \to N_c$, $\delta N_p \propto (N_c - N)^{-1/2}$.

In Fig. 2 we report δN_p as a function of temperature for some values of N, using the second term of Eq. (28). ϕ_1 and ϵ_1 have been evaluated with the IRAM in a lattice. We plot as a reference also the number of thermal atoms (dashed line) $N_{th} = \zeta(3)(kT/\hbar\omega_{ho})^3$ [26], where $\omega_{ho} \equiv (\omega_\parallel \omega_\perp^2)^{1/3}$ and

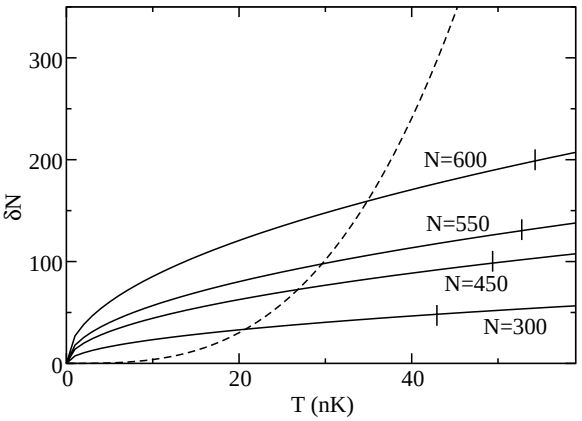

FIG. 2. Atomic fluctuation $(\delta N)_p$ as a function of the temperature for some values of N. Only the mode ϕ_1 is considered. The dashed line is the number of thermal atoms N_{th}. The vertical bars indicate the number of atoms for which $N_{th} = N$.

$\zeta(3)\approx 1.2$ is the Riemann zeta function of argument 3. The vertical bars indicate the number of atoms for which $N_{th}=N$. Our results are valid in the Bogoliubov approximation. As already said, in a harmonic trap the condensed particles form a high-density cloud with a low-density background of noncondensed particles, which we can neglect. However, since near the breaking point the collective density fluctuations become very large, the cubic and quartic terms neglected in the Hamiltonian may be important and the Bogoliubov approximation is no longer valid. The Hamiltonian of the system is

$$\mathcal{H}=\frac{1}{2}\int \hat{\psi}^{\dagger}H\hat{\psi}\,d^3x+\frac{g}{2}\int \hat{\psi}^{\dagger}\hat{\psi}^{\dagger}\hat{\psi}\hat{\psi}\,d^3x, \tag{30}$$

where H is the single-particle Hamiltonian. In the Heisenberg representation, the corresponding dynamical equation for $\hat{\psi}$ is

$$i\hbar\frac{\partial\hat{\psi}}{\partial t}=H\hat{\psi}+g\hat{\psi}^{\dagger}\hat{\psi}\hat{\psi}. \tag{31}$$

Let us perform the transformation (1), where ψ satisfy the time-dependent Gross-Pitaevskii equation, which can be obtained from Eq. (2) by replacing μ with $i\hbar\partial_t$. Then the equation for the field $\hat{\phi}$ is

$$i\hbar\frac{\partial\hat{\phi}}{\partial t}=H\hat{\phi}+g(2|\psi|^2\hat{\phi}+\psi^2\hat{\phi}^{\dagger})+g(\psi^*\hat{\phi}^2+2\hat{\phi}^{\dagger}\hat{\phi}^2). \tag{32}$$

In the Bogoliubov equation the last term is neglected. Its order of magnitude is $t_{nl}=g[\sqrt{n}\,\delta n+(\delta n)^{3/2}]\simeq\sqrt{n}\,\delta n$, where n and δn are the atomic density and its fluctuation. The order of magnitude of the second term is $t_l=n(\delta n)^{1/2}$. Quite in general, the nonlinear term in $\hat{\phi},\hat{\phi}^{\dagger}$ is negligible if it is small with respect to the interaction linear term, i.e., if

$$t_{nl}\ll t_l\rightarrow\delta n\ll n.$$

In our case, since the main contribution to the atomic fluctuation is given by δN_p, the Bogoliubov approximation is valid for

$$\delta N_p\ll N.$$

Using Eq. (28), this condition becomes

$$\left(\frac{kT}{2\epsilon_1 N}\right)^{1/2}\ll 1. \tag{33}$$

The eigenvalue ϵ_1 is of the order of $\hbar\omega_c$ [see Eq. (27)] and ω_c is of the order of $\omega_{\parallel}$ (in general, it is smaller). Thus, supposing that $\epsilon_1\simeq\hbar\omega_{\parallel}\simeq\hbar\omega_{\perp}$, we have

$$\frac{T}{T_t}\ll 4N^{1/3}, \tag{34}$$

where T_t is the critical temperature of condensation and, in absence of the sinusoidal potential, is equal to [26]

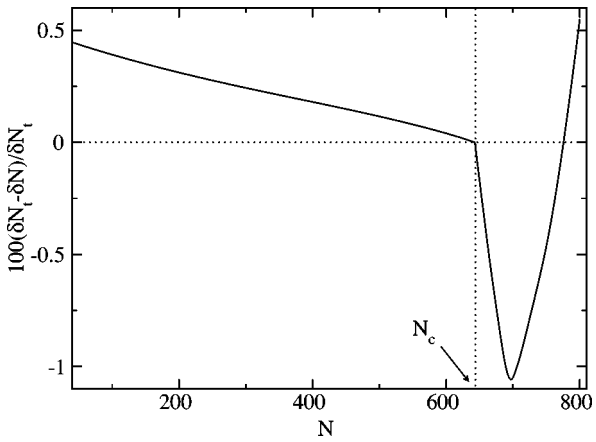

FIG. 3. Relative error $\mathcal{E}\equiv[\delta N-(\delta N)_p]/\delta N$ as a function of N.

$$T_t=0.94\hbar(\omega_{\parallel}\omega_{\perp}^2)^{1/3}N^{1/3}. \tag{35}$$

The condition (34) is satisfied also with $T=T_t$, i.e., it is, in general, always satisfied. However, Eq. (34) is optimistic, since near the symmetry breaking value N_c, the quantity ϵ_1 becomes quite smaller than $\hbar\omega_{\parallel}$, so the second term of Eq. (34) has to be multiplied by a number less than 1, i.e., T has to be smaller to have a sufficiently small δN_p. This is confirmed by Fig. 2. For $N=300$ and a temperature such that $N_{th}=N$, we have $\delta N_p\simeq 50=N/6$. Instead, for $N=N_{th}=600$ we have $\delta N_p\simeq 200=N/3$. For $N=N_{th}\sim N_c$ the quantity $\delta N_p/N$ can become greater than 1 and the Bogoliubov approximation is not valid. Thus, near N_c the temperature has to be sufficiently lower than T_t. Also when the collective fluctuations do not impose a pressing condition on T, the temperature has to be not too close to T_t, because of the presence of the thermal cloud. In our considerations, we have supposed that the collective mode temperature, which is related to δN_p, is equal to the thermal cloud temperature, i.e., that the system is at thermal equilibrium. However, these two values may be different (see the Introduction).

Note that the atomic fluctuations δN_p of the field are comparable to, or larger than, N_{th} for $T\leqslant 0.4$–$0.5T_t$. This is true also far from N_c. The collective atomic fluctuations can be experimentally observed by means of the phase-contrast imaging technique [27,28].

We have also evaluated the total fluctuations δN, i.e., with the whole spectrum of H_1, using the relaxation method. In Fig. 3 we report the relative error $\mathcal{E}\equiv[\delta N-(\delta N)_p]/\delta N$ as a function of N. δN includes all the eigenvectors ϕ_k. The figure shows that δN_p is a very good estimate of the atomic fluctuations. Note that above N_c the error is negative. This is due to ϕ_0, which contributes with a negative value to δN. Below N_c, ϕ_0 is nearly equal to ψ; thus its effect on $\mathcal{E}$ is negligible $(Q|\phi_0\rangle\approx 0)$. However, this is not the case slightly above N_c.

Before concluding this section, we want to discuss under what conditions the approximate equation (16) is valid. The ''exact'' δn_m is

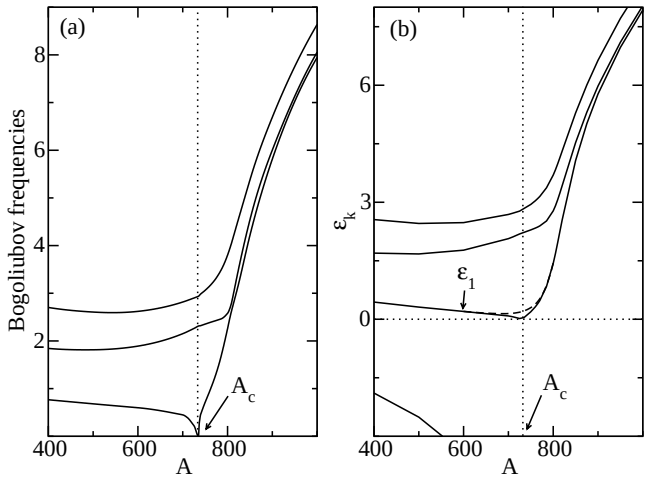

FIG. 4. (a) Lowest Bogoliubov frequencies and (b) ϵ_k as a function of A for a one-dimensional array of wells and a one-dimensional space. A_c is the critical point, where the symmetry breaking occurs. The dashed line near the ϵ_1 frequency is evaluated with $A<0$ (in the abscissa $|A|$ is plotted). In this case the frequency does not drop to zero at $A=A_c$.

$$4\int d^3x\, d^3y\, \Omega(\vec{x})\Omega(\vec{y})\psi(\vec{x})\psi(\vec{y}) \times [QH_1^{-1}(H_1H_2)^{1/2}A_T^{-1}Q](\vec{x},\vec{y}). \tag{36}$$

Let χ_k and f_k be the eigenvectors and eigenvalues of $(H_1H_2)^{1/2}$, we have that

$$\langle\vec{x}|A_T^{-1}|\vec{y}\rangle=\sum_k \tanh^{-1}(\beta f_k/2)\chi_k(\vec{x})\chi_k(\vec{y}). \tag{37}$$

Only the eigenvectors that have few nodes contribute to δn_m. Let f^M be their maximum eigenvalue; then Eq. (16) can be considered as valid when $\beta f^M/2\lesssim 1$, since $A_T^{-1}\simeq 2kT(H_1H_2)^{-1/2}$. In the case that we have considered, $\Omega(\vec{x})$ is 0 for $x_1<0$ or $x_1>0$; thus it is reasonable to suppose that $f^M\sim\hbar\omega_c$, where ω_c is the collective oscillation frequency of the condensate. We have the condition

$$kT>2\hbar\omega_c. \tag{38}$$

Equation (38) is always satisfied in the present experiments, since in the case of the double-well we have in general $\omega_c\lesssim\omega_\parallel$. In fact, when the symmetry breaking is not present, ω_c is the tunneling frequency between the two wells and it is lower than $\omega_\parallel$. This inequality is not valid in general above the symmetry breaking point, since the condensate tends to cluster in a single well and ω_c becomes the oscillation frequency inside the single well. This is confirmed by Fig. 4(a), which will be discussed in the following section. The lowest frequency corresponds to ω_c in the case of an array of wells. It decreases below the symmetry breaking point and is smaller than the trap frequency, which is equal to 1. Note that on the horizontal axis there is A (see next section), not N. For $A=0$ the lowest frequency is equal to the trap frequency. Anyway, we can consider Eq. (38) valid for a large range of the parameters.

In the following section we will verify numerically the validity of Eq. (16) in the case of a one-dimensional array of wells [see the inset of Fig. 6 below].

In conclusion, the Yukawa-like equation approach can be used when the Bogoliubov approximation is valid and a condition like Eq. (38) is satisfied. These two conditions establish a rather large spectrum of validity.

V. ONE-DIMENSIONAL ARRAY OF POTENTIAL WELLS

In this section we study an attractive condensate in a cigar-shaped potential plus a sinusoidal potential in the axial direction, showing that, as in the case of a double-well potential, a symmetry breaking can occur. To compare the results obtained by Eq. (13) with the ones obtained by Eq. (16), we first solve the problem numerically in one dimension. We evaluate both Bogoliubov frequencies and H_1 eigenvalues. Then we extend the calculations to the tridimensional problem.

In nondimensional units, the time-independent Gross-Pitaevskii is

$$\left[-\frac{1}{2}\frac{d^2}{dx^2}+V+g|\psi|^2\right]\psi=\mu\psi. \tag{39}$$

The potential V has the following form:

$$V(x)=\frac{1}{2}x^2+A\cos(kx). \tag{40}$$

$\psi(x)$ is normalized to unity, and thus g also contains the number of atoms N. The system displays spatial symmetry breaking, with formation of two stationary states, for $A>0$ and a suitable number of atoms. The bifurcation occurs, for a fixed A and increasing N, or for a fixed number of atoms and increasing A, i.e., the potential barriers. We study the process by continuously varying A, with $g=-1$ and $k=12\pi$. For $A>0$ and a low number of atoms, the ground state wave function has a minimum at the center of the condensate, for $A<0$, it has a maximum. Intermediate cases can be obtained by adding a constant phase ϕ_0 in the argument of the cosine function of the potential $V(x)$. When $\phi_0\neq n\pi/2$, where n is an integer, the potential is not symmetric, thus we have, not a symmetry breaking, but simply a possible bifurcation, i.e., the minimal energy state splits into two states with local minimal energy above/below a critical positive/negative value A_c of A. For $A>0$ or $A<0$, there exists an interval centered in $\phi_c=\pi/2$ or $\phi_c=0$, respectively, where only a local minimum of energy exists. Thus, for a suitable value of A, we can obtain a bifurcation by merely shifting the sinusoidal potential out of the above-mentioned interval.

In contrast with the previous case, we fixed the nonlinear term and changed A. In Fig. 4 we report some Bogoliubov frequencies [Fig. 4(a)] and ϵ_k [Fig. 4(b)] as a function of A. In both cases there is a frequency that drops to zero at the symmetry breaking critical point $A_c\simeq 735$. However, the be-

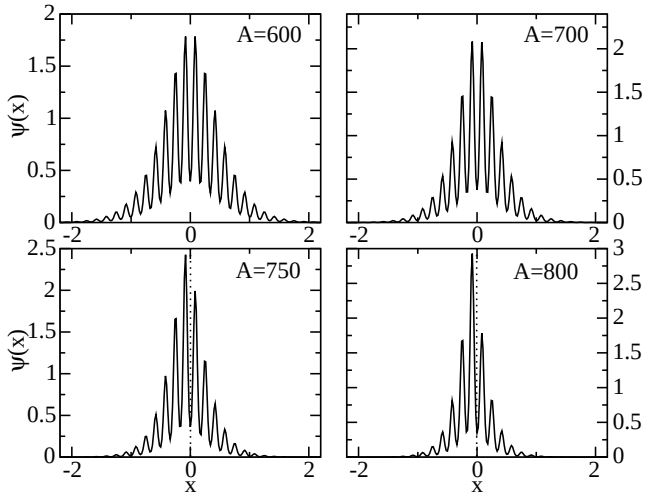

FIG. 5. Wave function in the case of a one-dimensional array for $k=12\pi$ and some values of A. Note the symmetry breaking for $A=750$ and $A=800$.

havior of this frequency is quite different in the two cases. Below A_c, ϵ_1 is nearly a linear function of A, in contrast with the first Bogoliubov frequency. The dashed line in Fig. 4(b) is ϵ_1 evaluated with $A<0$ (in this case, $|A|$ is plotted on the abscissa). Note that this frequency does not drop to zero at $A=A_c$, i.e., there is not a symmetry breaking. The atomic fluctuations are nearly proportional to $1/\epsilon_1$. In Fig. 5 we plot $\psi(x)$ below A_c ($A=600,700$) and above A_c ($A=750,800$). In this last case the spatial distribution becomes asymmetric. In Fig. 6 we plot $\delta N/N^{1/2}$ as a function of the temperature T in nondimensional units for some values of N, evaluated with the "exact" Eq. (13). δN is the fluctuation of the number of atoms in the region with $x>0$ (or, also, with $x<0$). We have chosen to report $\delta N/N^{1/2}$, since this quantity is unequivocally determined by the product Na, where a is the scattering length. In fact, by choosing g, we have fixed this product, but

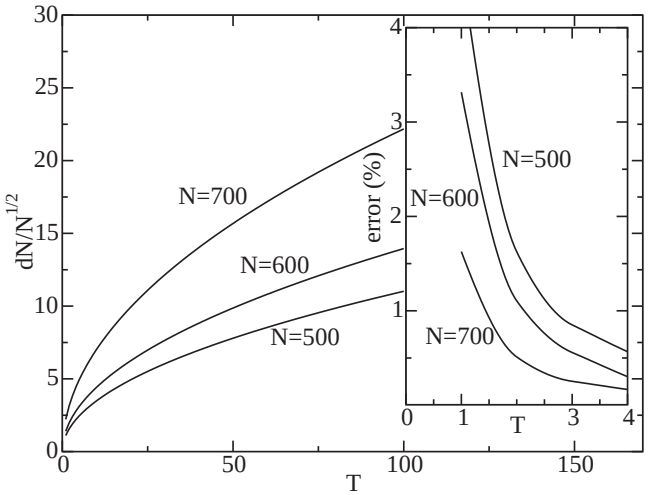

FIG. 6. $\delta N/N^{1/2}$ as a function of T in nondimensional units for some values of N. The function is evaluated using Eq. (13). In the inset the relative error $(\delta N-\delta N_{approx})/\delta N$ is reported.

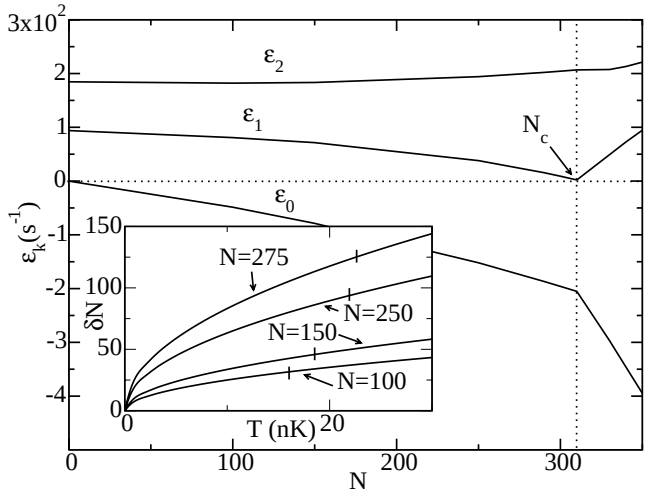

FIG. 7. ϵ_k as a function of N in the three-dimensional case of an array of wells. In the inset, the fluctuation δN is reported as a function of T for some values of N.

not N. In the inset we report the relative error $(\delta N-\delta N_{approx})/\delta N$, where δN_{approx} is the number fluctuation evaluated with the approximate Eq. (16). When $T\gg 1$ the error is very small, in accordance with the condition (38), since ω_c is of the order of the second Bogoliubov frequency, that is, ~ 2. Note that the error diminishes with increasing N.

Now we consider the problem in three dimensions. In practice, we refer to the same system as in the previous section, with the potential (26), but with $\omega_{\|}=2\pi\times 20\ \mathrm{s}^{-1}$ and $A=4000\ \mathrm{s}^{-1}$. In this case the harmonic trap is larger in the axial direction and thus there are more wells. We also increase A, because for the previous value, $2000\ \mathrm{s}^{-1}$, no symmetry breaking occurs. In Fig. 7 we report the first three eigenvalues ϵ_k as a function of N. The symmetry breaking occurs at $N\simeq 300$. The qualitative aspect is identical to the double-well case. In the inset we report the fluctuations of the number of atoms δN in a region with $x_1>0$ (or $x_1<0$) [see Eq. (26)] as a function of T and for some values of N. The vertical bars indicate the number of atoms for which $N_{th}=N$. Also in this case the fluctuations can be very large. The approach that uses the Yukawa-like equation is very useful in this case, where we have a three-dimensional system with a fine structure potential, which requires a discrete lattice with many points for the numerical calculations and, with the standard approach, we should have diagonalized a very large matrix. Also using the ARPACK library for evaluating a portion of the spectrum, our approach is more efficient, since we have to evaluate the lowest eigenvalue spectrum of the involved matrix, whereas in the case of the operator $\mathcal{L}$ the most significant spectrum is an internal band (see Sec. III B).

VI. CONCLUSION

We have shown that it is possible to use the Yukawa-like equation introduced in Ref. [17] to evaluate the atomic density fluctuations in a region inside a Bose-Einstein condensate. For suitable high temperatures we can replace the op-

eratorial source term in the Yukawa-like equation with a Dirac delta. We have applied the Yukawa-like equation to the case of a condensate of attractive atoms in a double-well potential to evaluate the density fluctuation δN inside a single well. This system undergoes a symmetry breaking above a critical number N_c of atoms. Near N_c, δN can be very large and observable experimentally. We find that the approximated source term can be used when the temperature is of the order of, or larger than, $\hbar\omega_c$, where ω_c is the collective oscillation frequency between the two wells. This condition is always satisfied in the present experiments.

δN is related to the collective mode temperature, which may be different from the temperature measured by means of the thermal tails, since the collective mode may be weakly coupled with the other ones. For this reason, an experimental measurement of δN might be interesting. A weak coupling is important to reduce the decoherence effects in the macroscopic quantum superposition experiments proposed in Refs. [19,21].

We have considered also the case of an array of wells, obtained with the superposition of a sinusoidal potential and a harmonic potential. We initially studied the one-dimensional case to verify the advisability of using the approximated source term. When at the center O of the harmonic potential there is a minimum or a maximum of the total potential, this last is symmetric with respect to O. We find that the symmetry breaking with respect to O occurs only when the potential has a maximum at the center of the trap. Finally, to underline the power of our approach, we considered a tridimensional potential with a fine structure, obtained by the superposition of a sinusoidal standing wave and a tridimensional harmonic potential with axial symmetry. The direction of the standing wave is the symmetry axis of the harmonic potential. The evaluation of the density fluctuations with the standard approach (see the end of Sec. II) for this problem should be very heavy. In this case also, we obtain a symmetry breaking with a suitable choice of the parameters. As in the double-well case, near the breaking point the density fluctuations can be very large and observable experimentally.

[1] J. Javanainen and S.M. Yoo, Phys. Rev. Lett. **76**, 161 (1996).
[2] W. Hoston and L. You, Phys. Rev. A **53**, 4254 (1996).
[3] M. Naraschewski, H. Wallis, A. Schenzle, J.I. Cirac, and P. Zoller, Phys. Rev. A **54**, 2185 (1996); J.I. Cirac, C.W. Gardiner, M. Naraschewski, and P. Zoller, *ibid.* **54**, R3714 (1996).
[4] T. Wong, M.J. Collett, and D.F. Walls, Phys. Rev. A **54**, R3718 (1996); H. Wallis, A. Röhrl, M. Naraschewski, and A. Schenzle, *ibid.* **55**, 2109 (1997).
[5] A. Röhrl, M. Naraschewski, A. Schenzle, and H. Wallis, Phys. Rev. Lett. **78**, 4143 (1997).
[6] Y. Castin and J. Dalibard, Phys. Rev. A **55**, 4330 (1997).
[7] M. Naraschewski and R.J. Glauber, Phys. Rev. A **59**, 4595 (1999).
[8] M.R. Andrews, C.G. Townsend, H.J. Miesner, D.S. Durfee, D.M. Kurn, and W. Ketterle, Science **275**, 637 (1997).
[9] I. Bloch, T.W. Hänsch, and T. Essilinger, Nature (London) **403**, 166 (2000).
[10] W. Ketterle and H. Miesner, Phys. Rev. A **56**, 3291 (1997).
[11] E.A. Burt, R.W. Ghrist, C.J. Myatt, M.J. Holland, E.A. Cornell, and C.E. Wieman, Phys. Rev. Lett. **79**, 337 (1997).
[12] D.S. Petrov *et al.*, Phys. Rev. Lett. **87**, 050404 (2001).
[13] S. Dettmer *et al.*, Phys. Rev. Lett. **87**, 160406 (2001).
[14] C. Orzel *et al.*, Science **291**, 2386 (2001).
[15] M. Greiner *et al.*, Nature (London) **415**, 39 (2002).
[16] K. Mølmer, Phys. Rev. A **58**, 566 (1998).
[17] A. Montina, Phys. Rev. A **67**, 043604 (2003).
[18] A. Montina and E. Arimondo, Eur. Phys. J. D **14**, 105 (2001).
[19] A. Montina and F.T. Arecchi, Phys. Rev. A **66**, 013605 (2002).
[20] J.I. Cirac *et al.*, Phys. Rev. A **57**, 1208 (1998).
[21] A. Montina and F.T. Arecchi, Phys. Rev. A **67**, 023616 (2003).
[22] N. Bogoliubov, J. Phys. (Moscow) **11**, 23 (1947).
[23] Y. Castin and R. Dum, Phys. Rev. A **57**, 3008 (1998).
[24] W. Krauth, Phys. Rev. Lett. **77**, 3695 (1996).
[25] A. Sinatra, Y. Castin, and C. Lobo, J. Mod. Opt. **47**, 2629 (2000).
[26] F. Dalfovo *et al.*, Rev. Mod. Phys. **71**, 463 (1999).
[27] M.R. Andrews *et al.*, Science **273**, 84 (1996).
[28] J.M. Gerton *et al.*, Nature (London) **408**, 692 (2000).