# Synchronization in Coupled and Free Chaotic Systems

F.T. Arecchi, R. Meucci, E. Allaria, and S. Boccaletti

## 1. Introduction

Global bifurcations in dynamical systems are of considerable interest because they can lead to the creation of chaotic behaviour [Hilborn, 1994]. Global bifurcations are to be distinguished from local bifurcations around an unstable periodic solution. Typically, they occur when a homoclinic point is created. A homoclinic point is an intersection point between the stable and the unstable manifold of a steady state saddle point $p$ on the Poincaré section of a, at least, 3D flow. The presence of a homoclinic point implies a complicated geometrical structure of both the stable and the unstable manifolds usally referred to as a homoclinic tangle. When a homoclinic tangle has developed, a trajfectory that comes close to the saddle point behaves in an erratic way, showing sensitivity to initial conditions.

Homoclinic chaos (Arneodo et al., 1985) represents a class of self-sustained chaotic oscillations that exhibit quite different behaviour as compared to phase coherent chaotic oscillations. Typically, these chaotic oscillators possess the structure of a saddle point $S$ embedded in the chaotic attractor. In a more complex system, the orbit may go close to a second singularity which stabilizes the trajectory away from the first one, even though the chaotic wandering around the first singularity remains unaltered. We are thus in presence of a "heteroclinic" connection and will show experimentally some interesting peculiarities of it. Homoclinic and heteroclinic chaos (HC) have received large consideration in many physical (Arecchi, Meucci, and Gadomski, 1987), chemical (Argoul, Arneodo, and Richetti, 1987), and biological systems (Hodgkin and Huxley, 1952; Feudel et al., 2000).

Homoclinic chaos has also been found in economic models, that is, in heterogeneous market models (Brock and Hommes, 1997, 1998; Foroni and Gardini, 2003; Chiarella, Dieci, and Gardini, 2001). The heterogeneity of expectations among traders introduces the key nonlinearity into these models.

The physical system investigated is characterized by the presence, in its phase space, of a saddle focus SF and a heteroclinic connection to a saddle node SN. Interesting features have been found when an external periodic perturbation or a small amount of noise is added. Noise plays a crucial role, affecting the dynamics near the saddle focus and inducing a regularization of the chaotic behaviour. This peculiar feature leads to the occurrence of a stochastic resonance effect when a noise term is added to a periodic modulation of a system parameter.

Besides the behaviour of a single chaotic oscillator, we have investigated the collective behaviour of a linear array of identical HC systems. Evidence of spatial synchronization over the whole length of the spatial domain is shown when the coupling strength is above a given threshold. The synchronization phenomena here investigated are relevant for their similarities and possible implications in different fields such as financial market synchronization, which is a complex issue not yet completely understood.

## 2. The physical system

The physical system consists of a single mode CO2 laser with a feedback proportional to the output intensity. Precisely, a detector converts the laser output intensity into a voltage signal, which is fed back through an amplifier to an intracavity electro-optic modulator, in order to control the amount of cavity losses. The average voltage on the modulator and the ripple around it are controlled by adjusting the bias and gain of the amplifier (Fig. 1).

These two control parameters can be adjusted in a range where the laser displays a regime of heteroclinic chaos characterized by the presence of large spikes almost equal in shape but erratically distributed in their time separation T.

The chaotic trajectories starting from a neighbourhood of SF leave it slowly along the unstable manifold and have a fast and close return along the stable manifold after a large excursion (spike) which approaches the stabilizing fixed point SN (Fig. 2). Thus a significant contraction region exists close to the stable manifold. Such a structure underlies spiking behaviour in many neuron (Hodgkin and Huxley, 1952; Feudel et al. 2000), chemical (Argoul, Arneodo, and Richetti, 1987) and laser (Arecchi, Meucci, and Gadomski, 1987) systems. It is important to note that these HC systems has intrinsically highly nonuniform dynamics and the sensitivity to small perturbations is high only in the vicinity of SF, along the unstable directions. A weak noise thus can influence T significantly.



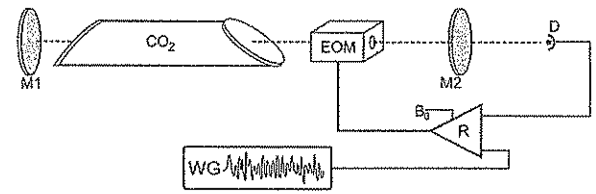


**Fig. 1.** Experimental setup. M1 and M2: mirrors; EOM: electro-optical modulator; D: diode detector; WG: arbitrary waveform and noise generator; R: amplifier; $B_0$: applied bias voltage

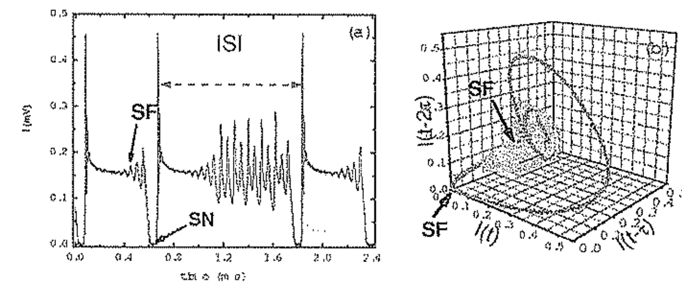


**Fig. 2.** Time evolution of the laser with feedback and the reconstructed attractor

## 3. Synchronization to an external forcing

The HC spikes can be easily synchronized with respect to a small periodic signal applied to a control parameter. Here, we provide evidence of such a synchronization when the modulation frequency is close to the "natural frequency", that is, to the average frequency of the return intervals. The required modulation is below 1%. An increased modulation amplitude up to 2% provides a wider synchronization domain which attracts a frequency range of 30% around the natural frequency. However, as we move away from the natural frequency, the synchronization is imperfect insofar as phase slips; that is, phase jumps of $\pm 2\pi$, appear.

Furthermore, applying a large negative detuning with respect to the natural frequency gives rise to synchronized bursts of homoclinic spikes separated by approximately the average period, but occurring in groups of 2, 3, etc. within the same modulation period (locking 1:2, 1:3 etc.) and with a wide inter-group separation. A similar phenomenon occurs for large positive detuning; this time the spikes repeat regularly every 2, 3 etc. periods (locking 2:1, 3:1 etc.) (Fig. 3).

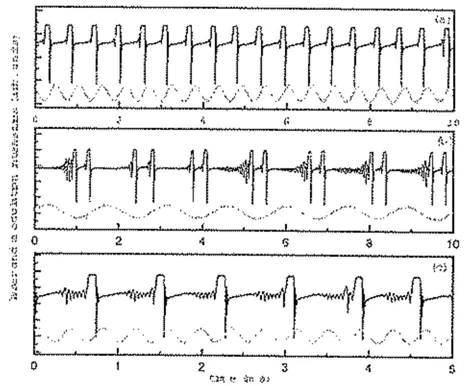

**Fig. 3.** Different response of the chaotic laser to external periodic modulation with different frequency with respect to the natural one of the system

The 1:1 synchronization regime has been characterized in the parameter space (amplitude and frequency) where synchronization occurs, Fig. 4. The boundary of the synchronization region is characterized by the occurrence of phase slips, where the phase difference from the laser signal and the external modulation presents a jump of $2\pi$ (grey dots).

## 4. Noise induced synchronization

Let us now consider the effects induced by noise. White noise is added as an additive perturbation to the bias voltage. From a general point of view, two identical systems which are not coupled, but are subjected to a common noise, can synchronize, as has been reported both in the periodic (Pikovsky, 1984; Jensen, 1998) and in the chaotic (Matsumoto and Tsuda, 1983; Pikovsky, 1992; Yu, Ott, and Chen, 1990) cases. For noise-induced synchronization (NIS) to occur, the largest Lyapunov exponent (LLE) ($\lambda_1 > 0$ in a chaotic system) has to become negative (Matsumoto and Tsuda, 1983; Pikovsky, 1992; Yu, Ott, and Chen 1990). However, whether noise can induce synchronization of chaotic systems has been a subject of intense controversy (Mritan and Banavaar, 1994; Pikovsky, 1994; Longa, Curado, and Oliveira, 1996; Herzel and Freund, 1995; Malescio, 1996; Gade and Basu, 1996; Sanchez, Matias, and Perez-Munuzuri, 1997).



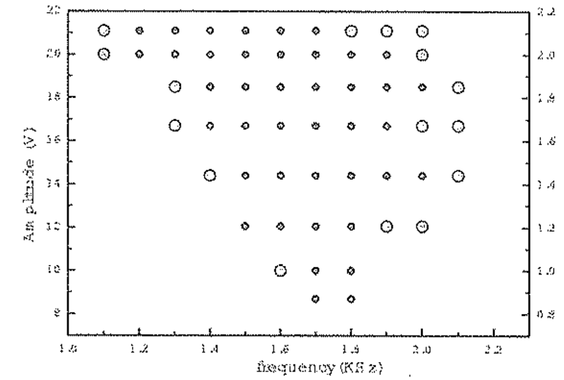

**Fig.4** Experimental reconstruction of the synchronization region for the parameters amplitude and frequency for the 1:1 synchronization regime

As introduced before, noise changes the competition between contraction and expansion, and synchronization ($\lambda_1 < 0$) occurs if the contraction becomes dominant.

We first carry out numerical simulations on the model which reproduces the dynamics of our HC system

$$\dot{x}_1 = k_0 x_1 \left(x_2 - 1 - k_1 \sin^2 x_6\right) \tag{1}$$

$$\dot{x}_2 = -\gamma_1 x_2 - 2k_0 x_1 x_2 + g x_3 + x_4 + p_0 \tag{2}$$

$$\dot{x}_3 = -\gamma_1 x_3 + g x_2 + x_5 + p_0 \tag{3}$$

$$\dot{x}_4 = -\gamma_2 x_4 + z x_2 + g x_5 + z p_0 \tag{4}$$

$$\dot{x}_5 = -\gamma_2 x_5 + z x_3 + g x_4 + z p_0 \tag{5}$$

$$\dot{x}_6 = -\beta\left(x_6 - b_0 + \frac{r x_1}{1 + \alpha x_1}\right) + D\xi(t) \tag{6}$$

In our case, $x_1$ represents the laser output intensity, $x_2$ the population inversion between the two resonant levels, $x_6$ the feedback voltage signal which controls the cavity losses, while $x_3$, $x_4$ and $x_5$ account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band. Furthermore, $k_0$ is the unperturbed cavity loss parameter, $k_1$ determines the modulation strength, $g$ is a coupling constant, $\gamma_1$, $\gamma_2$ are population relaxation rates, $p_0$ is the pump parameter, $z$ accounts for the number of rotational levels, and $\beta, r, \alpha$ are respectively the bandwidth, the amplification and the saturation factors of the feedback loop. With the following parameters $k_0 = 28.5714$, $k_1 = 4.5556$, $\gamma_1 = 10.0643$, $\gamma_2 = 1.0643$, $g = 0.05$, $p_0 = 0.016$, $z = 10$, $\beta = 0.4286$, $\alpha = 32.8767$, $r = 160$ and $b_0 = 0.1031$, the model reproduces the regime of homoclinic chaos observed experimentally (Pisarchik, Meucci, and Arecchi, 2001).

The model is integrated using a Heun algorithm (Garcia-Ojalvo and Sancho, 1999) with a small time step $\Delta t = 5 \cdot 10\text{-}5$ms (note that typical $T \approx 0.5$ ms).

Since noise spreads apart those points of the flow which were close to the unstable manifold, the degree of expansion is reduced. This changes the competition between contraction and expansion, and contraction may become dominant at large enough noise intensity $D$. To measure these changes, we calculate the largest Lyapunov exponent (LLE) $\lambda_1$ in the model as a function of $D$ (Fig. 5a, dotted line). $\lambda_1$ undergoes a transition from a positive to a negative value at $Dc \approx 0.0031$. Beyond Dc, two identical laser models x and y with different initial conditions but the same noisy driving $D\xi(t)$ achieve complete synchronization after a transient, as shown by the vanishing normalized synchronization error (Fig.5a, solid line).

$$E = \frac{\langle |x_1 - y_1| \rangle}{\langle |x_1 - \langle x_1 \rangle| \rangle} \qquad (7)$$

At larger noise intensities, expansion becomes again significant; the LLE increases and synchronization is lost when $\lambda_1$ becomes positive for $D$=0.052. Notice that even when $\lambda_1 < 0$, the trajectories still have access to the expansion region where small distances between them grow temporally. As a result, when the systems are subjected to additional perturbations, synchronization is lost intermittently, especially for $D$ close to the critical values. Actually, in the experimental laser system, there exists also a small intrinsic noise source. To take into account this intrinsic noise in real systems, we introduce into the equations $x_6$ an equivalent amount of independent noise (with intensity $D$=0.0005) in addition to the common one $D\xi(t)$. By comparison, it is evident that the sharp transition to a synchronized regime in fully identical $HC$ systems (Fig. 5a, solid line) is smeared out (Fig. 5a, closed circles).



In experimental study of NIS, for each noise intensity $D$ we repeat the experiment twice with the same external noise. Consistently with numerical results with a small independent noise, E does not reach zero due to the intrinsic noise, and it increases slightly at large $D$ (Fig. 5b).

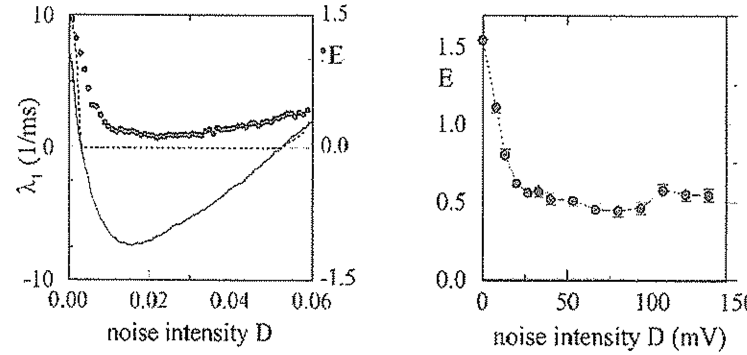


**Fig. 5.** Lyapunov exponents calculated from the model of the laser system and the synchronization error form the simulations and experimental showing the range of noise providing the synchronization between two systems

Experimental evidence of the noise induced synchronization is also provided by the data reported in Fig. 6.

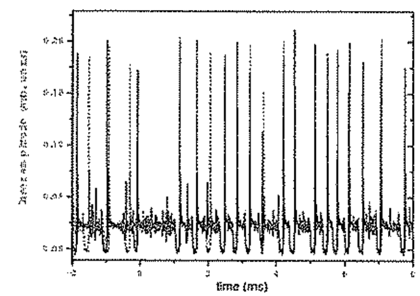

**Fig. 6.** Experimental evidence of noise induced synchronization, the action of the common noise signal starts at time $t$=0

## 5. Coherence Resonance and Stochastic Resonance

Coherence effects have been investigated by considering the distribution of the return time $T$ which is strongly affected by the presence of noise. These effects are more evident in the model than in the experiment.

The model displays a broad range of time scales. There are many peaks in the distribution $P(T)$ of $T$, as shown in Fig. 7a. In the presence of noise, the trajectory on average cannot come closer to $SF$ than the noise level. As a consequence the system spends a shorter time close to the unstable manifold. A small noise ($D$=0.0005) changes significantly the time scales of the model: $P(T)$ is now characterized by a dominant peak followed by a few exponentially decaying ones (Fig. 7b).

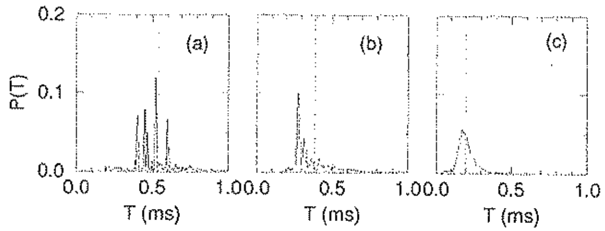


**Fig. 7.** Noise-induced changes in time-scales. (a) $D$=0; (b) $D$=0.0005; and (c) $D$=0.01. The dotted lines show the mean interspike interval $T_0(D)$, which decreases with increasing $D$

This distribution of $T$ is typical for small $D$ in the range $D = 0.00005 \div 0.002$. The experimental system with only intrinsic noise (equivalent to $D = 0.0005$ in the model) has a very similar distribution $P(T)$ (not shown). At larger noise intensity $D = 0.01$, the fine structure of the peaks is smeared out and $P(T)$ becomes an unimodal peak in a smaller range (Fig. 7c). Note that the mean value $T_0(D)=\langle T \rangle t$ (Fig. 7, dotted lines) decreases with $D$. When the noise is rather large, it affects the dynamics not only close to $S$ but also during the spiking, so that the spike sequence becomes fairly noisy. We observe the most coherent spike sequences at a certain intermediate noise intensity, where the system takes a much shorter time to escape from $S$ after the fast reinjection, the main structure of the spike being preserved.

As a result of noise-induced changes in time scales, the system displays a different response to a weak signal ($A$ = 0.01) with a frequency

$$f_e = f_0(D) = \frac{1}{T_0(D)},$$

i.e., equal to the average spiking rate of the unforced model. At $D = 0$, $P(T)$ of the forced model still has many peaks (Fig. 8a), while at $D = 0.0005$, T is sharply distributed around the signal period $T_e = T_0(D)$ (Fig. 8b). However, at larger intensity $D$=0.01, $P(T)$ becomes lower and broader again (Fig. 8c).



In the model and experimental systems, the pump parameter $p_0$ is now modulated as

$$p(t) = p_0[1 + A \cdot \sin(2\pi f_e t)] \tag{8}$$

by a periodic signal with a small amplitude $A$ and a frequency $f_e$.

First we focus on the constructive effects of noise on phase synchronization. To examine phase synchronization due to the driving signal, we compute the phase difference $\theta(t)=\phi(t)-2\pi f_e t$. Here the phase $\phi(t)$ of the laser spike sequence is simply defined as [Pikovsky et al. ]

$$\phi(t) = 2\pi\left(k + \frac{t-\tau_k}{\tau_{k+1}-\tau_k}\right)(\tau_k \le t \le \tau_{k+1}) \tag{9}$$

where $\tau_k$ is the spiking time of the kth spike.

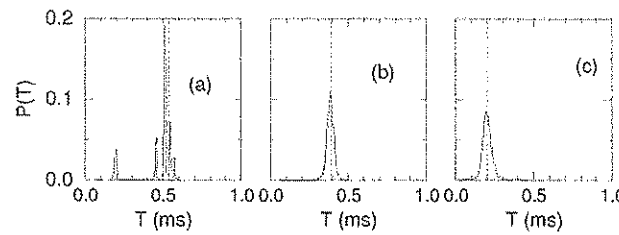


**Fig. 8.** Response of the laser model to a weak signal (A=0.01) at various noise intensities. (a) D=0; (b) D=0.0005; and (c) D=0.01. The signal period $T_e$ in (a), (b), and (c) corresponds to the mean interspike interval $T_0$(D) of the unforced model (A=0)

We have investigated the synchronization region (1:1 response) of the laser model in the parameter space of the driving amplitude A and the relative initial frequency difference $\Delta\omega = \frac{|f_e - f_o(D)|}{f_o(D)}$, where the average frequency $f_o(D)$ of the unforced laser model is an increasing function of $D$. The actual relative frequency difference in the presence of the signal is calculated as $\Delta\Omega = \frac{f - f_e}{f_o(D)}$ where $f = \frac{1}{\langle T \rangle_t}$ is the average spiking frequency of the forced laser model. The synchronization behaviour of the noise-free model is quite complicated and featureless at weak forcing amplitudes (about $A$<0.012). As shown in Fig. 9a, there does not exist a tonguelike region similar to the Arnold tongue in phase-coherent oscillators; for a fixed $A$, $\Delta\Omega$ is not a monotonous function of $\Delta\omega$ and it vanishes only at some specific signal frequencies; at stronger driving amplitudes about ($A > 0.012$), the system becomes periodic at a large frequency range. The addition of a small noise, $D = 0.0005$, drastically changes the response: a tonguelike region Fig. 9b, where effective frequency locking ($\|\Delta\Omega\| \leq 0.003$) occurs, can be observed similar to that in usual noisy phase-coherent oscillators. The synchronization region shrinks at a stronger noise intensity $D = 0.005$ (Fig. 9c).

The very complicated and unusual response to a weak driving signal in the noise-free model has not been observed in the experimental system due to the intrinsic noise whose intensity is equivalent to $D = 0.0005$ in the model.

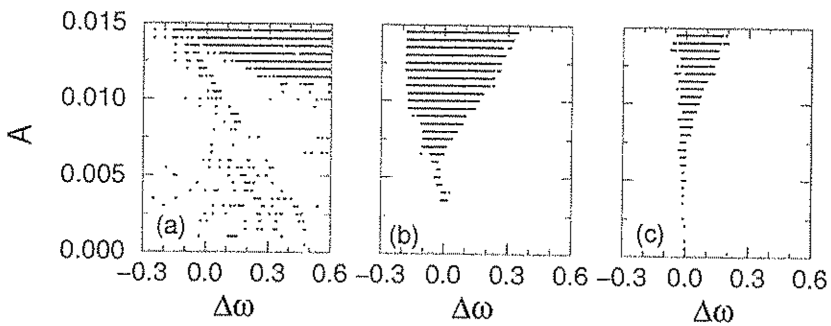


**Fig. 9.** Synchronization region of the laser model at various noise intensities. A dot is plotted when $|\Delta\Omega| \leq 0.003$. (a) $D=0$; (b) $D=0.0005$; and (c) $D=0.005$

We now study how the response is affected by noise intensity $D$ for a fixed signal period $T_e$. Here, in the chaotic lasers without the periodic forcing the average interspike interval $T_0(D)$ decreases with increasing $D$, and stochastic resonance (SR) similar to that in bistable or excitable can also be observed. We employ the following measure of coherence as an indicator of SR (Marino et al., 2002).

$$I = \frac{T_e}{\sigma_T} \int_{(1-\alpha)T_e}^{(1+\alpha)T_e} P(T)\,dT, \qquad (10)$$

where $0 < \alpha < 0.25$ is a free parameter. This indicator takes into account both the fraction of spikes with an interval roughly equal to the forcing period $T_e$ and the jitter between spikes [Marino et al. 2002]. SR of the 1:1 response to the driving signal has been demonstrated both in the model and in the experimental systems by the ratio $\langle T \rangle / T_e$ and I in Fig. 10. Again, the behaviour agrees well in the two systems. For $T_e < T_0(0)$, there exists a



synchronization region where $\langle T \rangle / T_e \approx 1$. The noise intensity optimizing the coherence I is smaller than the one that induces coincidence of T0(D) and $T_e$ (dashed lines in Fig. 10a, c). It turns out that maximal I occurs when the dominant peak of $P(T)$ (Fig. 9b) is located at $T_e$. This kind of noise-induced synchronization has not been reported in usual stochastic resonance systems, where at large $T_e$ numerous firings occur randomly per signal period and result in an exponential background in P(T) of the forced system, while at small $T_e$ a 1:n response may occur which means an aperiodic firing sequence with one spike for n driving periods on average (Marino et al. 2002; Benzi, Sutera, and Vulpiani 1981; Wiesenfeld and Moses 1995; Gammaitoni et al. 1998; Longtin and Chiaivo 1998).

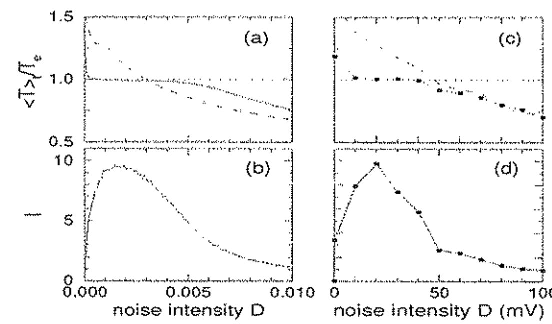


**Fig. 10.** Stochastic resonance for a fixed driving period. Left panel: model, $A = 0.01$, $T_e = 0.3$ *ms*. Right panel: experiments: forcing amplitude 10 *mV* ($A = 0.01$) and period, $T_e = 1.12$ *ms;* here the intensity $D$ is of the added external noise. Upper panel: noise-induced coincidence of average time scales (dashed line, $A = 0$) and synchronization region. Lower panel: coherence of the laser output. $\alpha = 0.1$ in Eq. (12)

## 6. Collective behaviour

Synchronization among coupled oscillators is one of the fundamental phenomena occurring in nonlinear dynamics. It has been largely investigated in biological and chemical systems.

Here we consider a chain of nearest neighbour coupled HC systems with the aim of investigating the emergence of collective behaviour by increasing the strength of the coupling [Leyva et al. 2003]. Precisely, going back to the model Eqs. (1)-(6) we add a superscript i to denote the site index. Then in the last equation for $x_6^i$, we replace $x_1^i$ with

$(1+\varepsilon)x_1^i + \varepsilon\left(x_1^{i+1} + x_1^{i-1} - 2\cdot\langle x_1^i\rangle\right)$ where $\varepsilon$ is a mutual coupling coefficient and $\langle x_1^i\rangle$ denotes a running time average of $x_1^i$.

We report in Fig. 11 the transition from unsynchronized to synchronized regimes by showing the space time representation of the evolution of the array. The transition to phase synchronization is anticipated by regimes where clusters of oscillators spike quasi-simultaneously (Leyva et al., 2003; Zheng, G. Hu, and B. Hu, 1998). The number of oscillators in the clusters increases with $\varepsilon$ extending eventually to the whole system (see Fig. 11c, d).

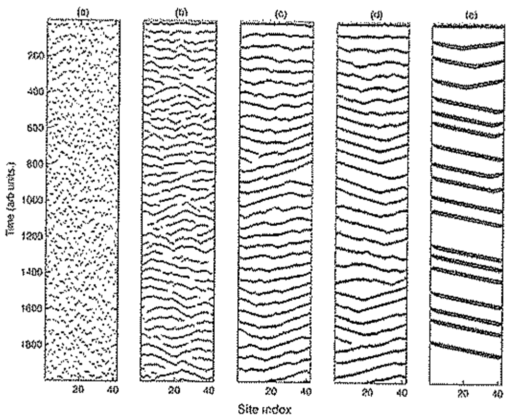


**Fig. 11.** Space-time representation of a chain of coupled chaotic oscillators for different values of $\varepsilon$ : (a) 0.0 , (b) 0.05, (c) 0.1 (d) 0.12 (e) 0.25

A better characterization of the synchronized pattern can be gathered by studying the emerging "defects" (Leyva et al., 2003). Each defect consists of a phase slip; that is a spike that does not induce another spike in its immediate neighbourhood. In order to detect the presence of defects, we map the phase in the interval between two successive spikes of a same oscillator as a closed circle $(0,2\pi)$, as usual in pulsing systems (For a review of the subject, see Pikovsky, Rosenblum, and Kurths 2001; Boccaletti et al.2002). Notice that since a suitable observer threshold isolates the spikes getting rid of the chaotic small inter-spike background, we care only for spike correlations. With this notation, we will consider that a defect has occurred when the phase of an oscillator has change by $2\pi$ while the phase of an immediate neighbour has changed by $4\pi$ . Here, our "phase synchronization" term denotes a connected line from left to right, which does not necessarily imply equal time occurrence of spikes at different



sites; indeed, the wavy unbroken lines which represent the prevailing trend as we increase $\varepsilon$ are what we call "phase synchronization".

## 7. Evidence of homoclinic chaos in financial market problems with heterogeneous agents

Important changes have influenced economic modelling during the last decades. Nonlinearities and noise effects have played a crucial role to explain the occasionally high irregular movements in economic and financial time series (Day 1994; Mantegna and Stanley, 1999).

One of the peculiar differences between economics and other sciences is the role of expectations or beliefs in the decisions of agents operating on markets. In rational expectation models, largely used in classic economy [Muth 1961], agents evaluate their expectations on the knowledge of the linear market equilibrium equations. Nowadays the rational equilibrium hypothesis is considered unrealistic and a growing interest is devoted to bounded rationality where agents are using different learning algorithms to predict their beliefs. In the last decade several heterogeneous market models have been introduced where at least two types of agents coexist (Brock and Hommes, 1997; 1998; Arthur, Lane, and Durlauf, 1997; Lux and Marchesi, 1999; Farmer and Joshi, 2002). The first group is composed of fundamentalists who believe that the asset prices are completely determined by economic fundamentals. The other group is composed of chartists or technical traders believing that the asset prices are not determined by their fundamentals but can be derived using trading rules based on previous prices. Brock and Hommes first discovered that a heterogeneous agent model with rational versus naive expectation can exhibit complicated price dynamics when the intensity of choice to switch between strategies is high (Brock and Hommes, 1997, 1998). Agents can either adopt a sophisticated predictor H1 (rational expectation) or another simple and low cost predictor H2 (adaptive short memory or naive expectation). Near the steady state or equilibrium price most of the agents use the naive predictor. Prices are driven far from the equilibrium. However, when prices diverge from their equilibrium, forecasting errors tend to increase and as a consequence it becomes more convenient to switch to the sophisticated predictor. The prices will move back to the equilibrium price. According to Brock and Hommes a heterogeneous market can be considered as a complex adaptive system whose dynamics are ruled by a two dimensional map for the variables $p_t$ and $m_t$ :

$$p_{t+1} = \frac{-b(1-m_t)p_t}{2B + b(1-m_t)} = f(p_t, m_t) \qquad (11)$$

$$m_{t+1} = \tanh\left(\frac{\beta}{2}\left(\frac{b}{2}\left(\frac{b(1-m_t)}{2B + b(m_t+1)} + 1\right)^2 p_t^2 - C\right)\right) = g(p_t, m_t) \qquad (12)$$

pt represents the deviations from the steady state price $p$ determined by the intersection of demand and supply. mt is the difference between the two fractions of agents, $\beta$ is the intensity of choice indicating how fast agents switch predictors. The parameter C is the price difference between the two predictors. The parameter b is related to a linear supply curve derived from a quadratic cost function $c(q) = q^2/2b$ where $q$ is the quantity of a given non-storable good.

The temporal evolution of the variables m and $p$ at the onset of chaos is reported in Fig. 12. The corresponding chaotic attractor is shown in Fig. 13. This attractor occurs after the merging of the two coexisting 4 piece chaotic attractors. The importance of homoclinic bifurcations leading to chaotic dynamics in economic models with heterogeneous agents has been recently emphasized by I. Foroni and L. Gardini, who extended the theoretical investigations also to noninvertible maps (Foroni and Gardini, 2003; Chiarella, Dieci, and Gardini, 2001).

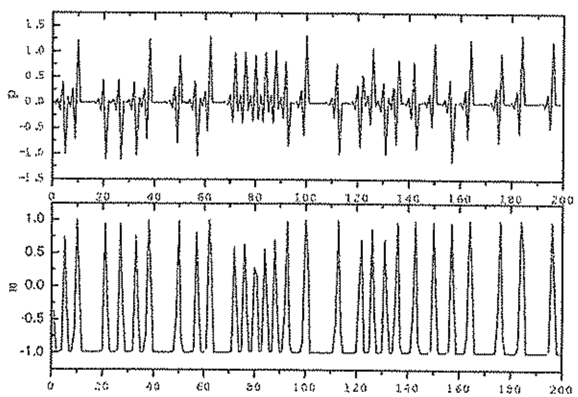

**Fig. 12.** Simulation of the model showing the chaotic evolution of the price $p$ and the agents difference $m$. The used parameters are $b = 1.35$, $B = 0.5$, $C = 1$ and $\beta = 5.3$



## 8. Conclusions

Heterogeneous markets models have received much attention during the last decade for the richness of their dynamical behaviours including homoclinic bifurcations to chaos as pointed out by the simple evolutionary economic model proposed by Brock and Hommes. This is similar to the behaviour of a laser with electro-optic feedback. The feedback process, acting on the same time scale of the other two relevant variables of the system, that is, laser intensity and population inversion, is the crucial element leading to heteroclinic chaos. In such a system, chaos shows interesting features. It can be easily synchronized with respect to small periodic perturbations and, in the presence of noise, it displays several constructive effects such as stochastic resonance.

As we have stressed in this paper, homoclinic/heteroclinic chaos is a common feature in many different disciplines including economics. Quite frequently, chaos is harmful, so controlling or suppressing its presence has been widely considered in recent years. However, in some situations chaos is a beneficial behaviour. Typically a chaotic regime disappears as a result of a crisis which is an abrupt change from chaos to a periodic orbit or a steady state solution. This occurrence finds its analogy in microeconomics when a firm is near bankruptcy. In such a critical condition, suitable and careful interventions are necessary in order to recover the usual cycle of the firm.

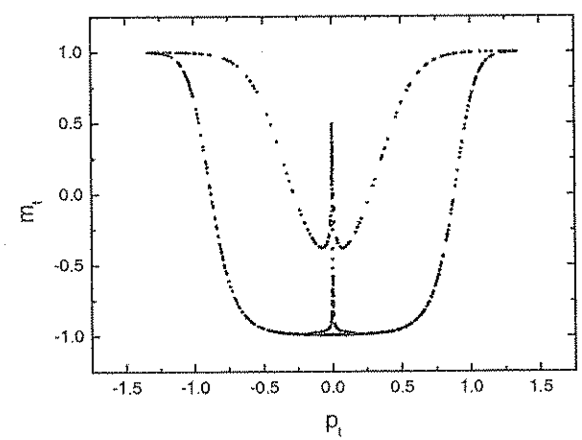


**Fig. 13.** Attractor of the dynamics of the model. The values of the parameter are the same reported in Fig. 12

Other important aspects concern the occurrence of synchronization phenomena in economics. Currently, the attention is on synchronization among the world's economies considering the greater financial openness. In this way, globalization effects lead to an increase of the links between the world's economies, in particular by means of capital markets and trade flows.

*Authors acknowledge MIUR-FIRB contract n. RBAU01B49F_002*

## References

Arecchi, F.T., R. Meucci, and W. Gadomski, (1987): "Laser Dynamics with Competing Instabilities", Phys. Rev. Lett. 58, 2205.

Argoul, F., A. Arneodo, and P. Richetti, (1987): "A propensity criterion for networking in an array of coupled chaotic systems", Phys. Lett. 120A, 269.

Arneodo, A., P.H. Coullet, E.A. Spiegel, and C. Tresser, (1985): "Asymptotic chaos", Physica D 14, 327.

Arthur, W., D. Lane, and S. Durlauf (eds.), (1997): *The Economy as an Evolving Complex System II*, Addison-Wesley, Redwood City, CA.

Benzi, R., A. Sutera, and A. Vulpiani, (1981): "The mechanism of stochastic resonance", *J. Phys. A* 14, L453.

Boccaletti, S., J. Kurths, G. Osipov, D. Valladares, and C. Zhou, (2002): "The synchronization of chaotic systems", Phys. Rep. 366, 1.

Brock, W.A., and C.H. Hommes, (1997): "A rational route to randomness", *Econometrica*, 65, 1059.

Brock, W.A., and C.H. Hommes, (1998): "Heterogeneous beliefs and routes to chaos in a simple asset pricing model", *Journ. of Econom. Dynam. and Control*, 22, 1235.

Chiarella, C., R. Dieci, and L. Gardini, (2001): "Asset price dynamics in a financial market with fundamentalists and chartists", *Discrete Dyn. Nat. Soc.* 6, 69.

Day, R.H., (1994): *Complex Economics Dynamics*, MIT Press, Cambridge, MA.

Farmer, J.D., and S. Joshi, (2002): "The Price Dynamics of Common Trading Strategies", *Journ. of Econ. Behavior and Organization* 49, 149.

Feudel, U. et al., (2000): "Homoclinic bifurcation in a Hodgkin-Huxley Model of Thermally Sensitive Neurons", *Chaos* 10, 231.

Foroni, I., and L. Gardini, (2003): "Homoclinic bifurcations in Heterogeneous Market Models", *Chaos, Solitons and Fractals* 15, 743-760.

Gade, P.M., and C. Basu, (1996): "The origin of non-chaotic behavior in identically driven systems", *Phys. Lett. A* 217, 21.

Gammaitoni L., P. Hanggi, P. Jung, and F. Marchesoni, (1998): "Stochastic Resonance", *Rev. Mod. Phys.* 70, 223.

Garcia-Ojalvo, J., and J.M. Sancho, (1999): *Noise in Spatially Extended System*, Springer, New York.



Herzel, H., and J. Freund, (1995): "Chaos, noise, and synchronization reconsidered", *Phys. Rev. E* 52, 3238.

Hilborn, R.C., (1994): *Chaos and Nonlinear Dynamics*, Oxford University Press, Oxford.

Hodgkin, A.L., and A.F. Huxley, (1952): "A quantitative description of membrane current and its application to conduction and excitation in nerve", *J. Physiol.* 117, 500.

Jensen, R.V., (1998): "Synchronization of randomly driven nonlinear oscillators", *Phys. Rev. E* 58, R6907.

Leyva, I., E. Allaria, S. Boccaletti, and F. T. Arecchi, (2003): "Competition of synchronization domains in arrays of chaotic homoclinic systems", *Phys. Rev. E* 68, 066209.

Longa, L., E.M.F. Curado, and A. Oliveira, (1996): "Roundoff-induced coalescence of chaotic trajectories", *Phys. Rev. E* 54, R2201.

Longtin, A., and D. Chialvo, (1998): "Stochastic and Deterministic Resonances for Excitable Systems", *Phys. Rev. Lett.* 81, 4012.

Lux, T., and M. Marchesi, (1999): "Scaling and criticality in a stochastic multiagent model of a financial market", *Nature* 397, 498.

Malescio, G., (1996): "Noise and Synchronization in chaotic systems", *Phys. Rev.* 53, 6551.

Marino, F. et al., (2002): "Experimental Evidence of Stochastic Resonance in an Excitable Optical System", *Phys. Rev. Lett.* 88, 040601.

Mantegna, R.N., and H.E. Stanley, (1999): *An Introduction to Econophysics: Correlations and Complexity in Finance*, Cambridge University Press.

Matsumoto, K., and I. Tsuda, (1983): "Noise-induced Order", J. Stat. Phys. 31, 87.

Maritan, A., and J.R. Banavar, (1994): "Chaos, Noise and Synchronization", *Phys. Rev. Lett.* 72, 1451.

Muth, J.F, (1961): "Rational Expectations and the Theory of Price Movements" *Econometrica* 29, 315.

Pikovsky, A., (1994): "Comment on "Chaos, Noise and Synchronization"", *Phys. Rev. Lett.* 73, 2931 .

Pikovsky, A., M. Rosenblum, and J. Kurths, (2001): *Synchronization: A Universal Concept in Nonlinear Sciences*, Cambridge, University Press.

Pikovsky, A.S., Radiophys, (1984): "Synchronization and stochastization of an ensemble of self-oscillators by external noise", *Quantum Electron.* 27, 576.

Pikovsky, A.S., M.G. Rosenblum, G.V. Osipov, and J. Kurths, (1997): "Phase Synchronization of Chaotic Oscillators by External Driving", *Physica D* 104.

Pisarchik, A.N., R. Meucci, and F.T. Arecchi, (2001): "Theoretical and experimental study of discrete behavior of Shilnikov chaos in a $CO_2$ laser", *Eur. Phys. J. D* 13, 385.

Sanchez, E., M.A. Matias, and V. Perez-Munuzuri, (1997): "Analysis of synchronization of chaotic systems by noise: An experimental study", *Phys. Rev. E* 56, 4068.

Wiesenfeld, K., and F. Moss, (1995): "Stochastic resonance and the benefits of noise: from ice ages to crayfish and SQUIDs", *Nature* 373, 33.

Zheng, Z., G. Hu, and B. Hu, (1998): "Phase Slips and Phase Synchronization of Coupled Oscillators", *Phys. Rev. Lett.* 81, 5318.