SELF-CONSISTENT LIGHT PROPAGATION IN A RESONANT MEDIUM *

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We study the undistorted motion of light pulses in a resonant medium and give the conditions for self-induced transparency (2π-pulse).

The propagation of a light pulse in a resonant lossless medium in its ground state has been treated analytically only in particular cases [1,2] or by computer techniques [3]. On the other hand the propagation in an amplifying lossy medium has received a thorough analytical treatment [4,5]. In this letter we give a general analytical treatment of the former problem, showing that, for a homogeneously broadened transition, the 2π-pulse is the only one which propagates with a constant shape at a constant envelope velocity $v < c$ (c = phase velocity in the host medium). We also show under what condition the same result holds for an inhomogeneous line.

The atoms-field interaction in a rotating-wave

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representation is described by the following quasi-linear system of equations [4]:

$$\frac{\partial}{\partial t}\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} pE_0R_3 - (\delta + \partial\psi/\partial t)R_2 \\ (\delta + \partial\psi/\partial t)R_1 \\ -pE_0R_1 \end{pmatrix} \quad (1)$$

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)\begin{pmatrix} E_0 \\ \psi \end{pmatrix} = \frac{\hbar\omega pN}{2\epsilon}\begin{pmatrix} \int_{-\infty}^{\infty} R_1 g(\delta)\,\mathrm{d}\delta \\ \int_{-\infty}^{\infty} R_2 g(\delta)\mathrm{d}\delta/E_0 \end{pmatrix}, \quad (2)$$

where the optical field and macroscopic polarization per unit volume and frequency are given by $E(z,t) = E_0(z,t)\cos(\kappa z - \omega t + \psi(z,t))$, $P(z,t,\delta) = N\hbar p[R_1(z,t,\delta)\sin(\kappa z - \omega t + \psi) + R_2(z,t,\delta) \times \cos(\kappa z - \omega t + \psi)]$, respectively; $R_3(z,t,\delta)$ is the difference between occupation probability of lower and upper states per unit frequency, N the atom density, $\hbar p$ the matrix element of the transition, ϵ the dielectric constant of the host medium, ω and ω_0 the center frequencies of the field spectrum and of the inhomogeneous atomic line $g(\delta)$ ($\delta = \omega - \omega'$ being the detuning with respect to the field). $g(\delta)$ is assumed symmetric around ω_0. Boundary conditions specifying the preparation of the atoms at some initial time: $R_i(\delta, z, t = -\infty) = R_{i0}(\delta)$ and the input field: $E_0(t, z=0) = E_0(t,0)$, $\psi(t, z=0) = 0$, uniquely characterize the solution [6]. In the case $\omega = \omega_0$ one can verify that the solution has $\psi \equiv 0$ everywhere.

We look for a pulse moving undistortedly with a velocity v. Changing to new variables $\tau = t + - z/v$, $\eta = z/v$, and putting the η-derivatives equal to zero, the above equations reduce to:

$$\frac{\mathrm{d}}{\mathrm{d}\tau}\begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} pE_0R_3 - \delta R_2 \\ \delta R_1 \\ -pE_0R_1 \end{pmatrix} \quad (3)$$

$$(1-c/v)\,\mathrm{d}E_0/\mathrm{d}\tau = \hbar\omega pN/(2\epsilon)\int_{-\infty}^{\infty} R_1(\delta,\tau)g(\delta)\mathrm{d}\delta\ . \quad (4)$$

The following integrals of motion exist

$$R_1^2 + R_2^2 + R_3^2 = A^2(\delta)\ ,$$

$$\epsilon(1-c/v)(\hbar\omega pN)^{-1}E_0^2 + \int_{-\infty}^{\infty} R_3 g(\delta)\,\mathrm{d}\delta = B\ , \quad (5)$$

$$E_0 = \hbar\omega pN[2\epsilon(1-c/v)]^{-1}\int_{-\infty}^{\infty} R_2\delta^{-1}g(\delta)\mathrm{d}\delta + C\ ,$$

where the constants are determined by the initial conditions. The first integral shows that the representative point (R_1, R_2, R_3) in an isospin space moves for any δ on a sphere, precessing around the vector sum of E_0 (aligned along axis 2) and δ (aligned along axis 3). Hence, at any time R_1 can be taken to differ from $R_{10} \equiv R_1$ ($\delta = 0, z, t$) for an even function of δ, $f(\delta)$, constant in time. Eq. (4) shows that R_{10} = const. $\times \dot{E}_0$ (dot means $\partial/\partial\tau$). Putting this into the second (6) we have

$$\dot{R}_2(\delta, z, t) = \delta f(\delta)R_{10} = \text{const.} \times \delta f(\delta)\dot{E}_0(z,t)\ . \quad (6)$$

By using the integrals (5) plus this relation the system (3), (4) reduces to the pendulum equation:

$$\dot{\varphi}^2 = D\cos\varphi + B \quad (7)$$

where

$$\dot{\varphi} = p\int_{-\infty}^{\tau} E_0\,\mathrm{d}\tau'\ ,$$

and

$$D = \hbar\omega pN[\epsilon(1-c/v)]^{-1}\int_{-\infty}^{\infty} A(\delta)g(\delta)\,\mathrm{d}\delta\ .$$

The solution of eq. (8) is given in terms of Jacobi elliptic functions with a parameter $m = 2D/(B+D)$, that is

$$E_0(\tau) = E_0\ \mathrm{dn}\ (\sqrt{B+D}\,\tau/2\,|\,m) \quad (8)$$

and associated τ dependences for R_1. It is easily seen from eq. (7) that $m = 0$ corresponds to no coupling (E_0 constant). If at $\tau = -\infty$ we assume no field present ($\dot{\varphi} = 0$) and the atoms prepared in the lower state ($\varphi = A$), then $m = 1$ and the above solution becomes:

$$E_0(\tau) = \sqrt{2D}/p\ \mathrm{sech}(\sqrt{\tfrac{1}{2}D}\ \tau)\ . \quad (9)$$

Putting this solution into the equations one finds the relation between pulse velocity v and width $\sqrt{2/D}$ given in ref. 1. The unbounded oscillatory behavior arising from eq. (8) for $1 < m < \infty$ corresponds to the excited medium put into a cavity, and can be correlated with the problem of self-pulsing in lasers [7-9].

For a homogeneous line, the pendulum equation (7) follows directly from the equations of motion, without postulating relation (6).

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