SELF-CONSISTENT LIGHT PROPAGATION IN A RESONANT MEDIUM *

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We study the undistorted motion of light pulses in a resonant medium and give the conditions for self-induced transparency (2$\pi$-pulse).

The propagation of a light pulse in a resonant lossless medium in its ground state has been treated analytically only in particular cases [1, 2] or by computer techniques [3]. On the other hand the propagation in an amplifying lossy medium has received a thorough analytical treatment [4, 5].

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In this letter we give a general analytical treatment of the former problem, showing that, for a homogeneously broadened transition, the 2$\pi$-pulse is the only one which propagates with a constant shape at a constant envelope velocity $v < c$ ($c =$ phase velocity in the host medium). We also show under what condition the same result holds for an inhomogeneous line.

The atoms-field interaction in a rotating-wave
representation is described by the following quasi-linear system of equations [4]:

\[
\frac{\partial}{\partial t} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} \rho E_0 R_3 - (\delta + \psi/\partial \delta) R_2 \\ (\delta + \psi/\partial \delta) R_1 \\ -\rho E_0 R_1 \end{pmatrix} \tag{1}
\]

\[
\left( \frac{\partial}{\partial t} + c \frac{\partial}{\partial z} \right) \begin{pmatrix} E_0 \\ \psi \end{pmatrix} = \frac{\hbar \omega p N}{2e} \int_{-\infty}^{\infty} R_1 g(\delta) \, d\delta + \int_{-\infty}^{\infty} R_2 g(\delta) \, d\delta/E_0, \tag{2}
\]

where the optical field and macroscopic polarization per unit volume and frequency are given by \( E(z, t) = E_0(z, t) \cos(\kappa z - \omega t + \psi(z, t)) \), \( P(z, t, \delta) = \hbar \rho R_1 \delta \sin(\kappa z - \omega t + \psi) + R_3 \cos(\kappa z - \omega t + \psi) \), respectively; \( R_3(z, t, \delta) \) is the difference between occupation probability of lower and upper states per unit frequency, \( N \) the atom density, \( \rho \) the matrix element of the transition, \( \kappa \) the dielectric constant of the host medium, \( \omega \) and \( \omega_0 \) the center frequencies of the field spectrum and of the inhomogeneous atomic line \( g(\delta) \) (\( \delta = \omega - \omega_0 \) being the detuning with respect to the field). \( g(\delta) \) is assumed symmetric around \( \omega_0 \).

Boundary conditions specifying the preparation of the atoms at some initial time: \( R_1(\delta, z, t = -\infty) = R_{10}(\delta) \) and the input field: \( E_0(t, z = 0) = E_0(t, 0), \psi(t, z = 0) = 0 \), uniquely characterize the solution [6]. In the case \( \omega = \omega_0 \) one can verify that the solution has \( \psi = 0 \) everywhere.

We look for a pulse moving undistortedly with a velocity \( v \). Changing to new variables \( \tau = t - z/v, \eta = z/v \), and putting the \( \eta \)-derivatives equal to zero, the above equations reduce to:

\[
\frac{d}{d\tau} \begin{pmatrix} R_1 \\ R_2 \\ R_3 \end{pmatrix} = \begin{pmatrix} \rho E_0 R_3 - \delta R_2 \\ \delta R_1 \\ -\rho E_0 R_1 \end{pmatrix} \tag{3}
\]

\[
(1-c/v) dE_0/d\tau = \hbar \omega p N/(2e) \int_{-\infty}^{\infty} R_1(0, \tau)g(\delta) \, d\delta. \tag{4}
\]

The following integrals of motion exist:

\[
R_1^2 + R_2^2 + R_3^2 = R_0^2(0),
\]

\[
\epsilon(1-c/v)(\hbar \omega p N)^{-1} E_0^2 = \int_{-\infty}^{\infty} R_3 g(\delta) \, d\delta = B, \tag{5}
\]

\[
E_0 = \hbar \omega p N[2\epsilon(1-c/v)]^{-1} \int_{-\infty}^{\infty} R_2^2 g(\delta) \, d\delta + C, \tag{6}
\]

where the constants are determined by the initial conditions. The first integral shows that the representative point \((R_1, R_2, R_3)\) in an isospin space moves for any \( \delta \) on a sphere, precessing around the vector sum of \( E_0 \) (aligned along axis 2) and \( \delta \) (aligned along axis 3). Hence, at any time \( R_1 \) can be taken to differ from \( R_{10} = R_1(\delta = 0, z, t) \) for an even function \( \delta, f(\delta) \), constant in time. Eq. (4) shows that \( R_{10} = \text{const.} \times B_0 \) (dot means \( \partial/\partial \tau \)). Putting this into the second (6) we have

\[
\dot{R}_2(0, z, t) = 0f(\delta) R_{10} = \text{const.} \times 5f(\delta) \dot{B}_0(0, t). \tag{7}
\]

By using the integrals (5) plus this relation the system (3), (4) reduces to the pendulum equation:

\[
\dot{\varphi}^2 = D \cos \varphi + B \tag{7}
\]

where

\[
\dot{\varphi} = \rho \int_{-\infty}^{\infty} E_0 \, d\tau, \tag{7}
\]

and

\[
D = \hbar \omega p N[e(1-c/v)]^{-1} \int_{-\infty}^{\infty} A(0) g(\delta) \, d\delta. \tag{7}
\]

The solution of eq. (8) is given in terms of Jacobi elliptic functions with a parameter \( m = 2D/(B + D) \), that is

\[
E_0(\tau) = E_0(0) \int_{-\infty}^{\tau} \sqrt{B + D} \, d\tau/2(m) \tag{8}
\]

and associated \( \tau \) dependences for \( R_1 \). It is easily seen from eq. (7) that \( \tau = 0 \) corresponds to no coupling \( (E_0 \text{ constant})\); if \( \tau = \infty \) we assume no field present \( (\varphi = 0) \) and the atoms prepared in the lower state \( (\varphi = A) \), then \( m = 1 \) and the above solution becomes:

\[
E_0(\tau) = \sqrt{2D/\rho} \tanh(\sqrt{2D/\rho} \tau). \tag{9}
\]

Putting this solution into the equations one finds the relation between pulse velocity \( v \) and width \( \sqrt{2/\rho} \) given in ref. 1. The unbounded oscillatory behavior arising from eq. (6) for \( 1 < m < \infty \) corresponds to the excited medium put into a cavity, and can be correlated with the problem of self-pulsing in lasers [7-9].

For a homogeneous line, the pendulum equation (7) follows directly from the equations of motion, without postulating relation (6).

3. F. A. Hopf and M. O. Scully, to be published.
7. H. Risken and K. Nummedal, to be published.