
ATOMIC COHERENT STATES IN QUANTUM OPTICS

F. T. Arecchi
Università di Pavia and C. I. S. E.
Milano, Italy

E. Courtens
IBM Research Laboratory
8803 Rüschlikon, Zurich, Switzerland

R. Gilmore*
Department of Physics
Massachusetts Institute of Technology
Cambridge, Massachusetts

H. Thomas
University of Frankfurt
Frankfurt, Germany

ABSTRACT

For the description of an assembly of two-level atoms, atomic coherent states can be defined which have properties analogous to those of the field coherent states. The derivation of the properties of the atomic coherent

*Present address: Department of Physics, University of South Florida, Tampa, Florida.
states is made easier by the use of a powerful disentangling theorem for exponential angular momentum operators. A complete labelling of the atomic states is developed and many of their properties are studied. In particular it is shown that the atomic coherent states are the quantum analogs of classical dipoles, and that they can be produced by classical fields.

I. INTRODUCTION

Many problems in quantum optics can be dealt with in terms of the interaction of an assembly of two-level atoms with a transverse electromagnetic field. In these problems a particular set of quantum states has to be selected for the description of both field and atoms. The choice of a particular representation is always motivated by convenience rather than by necessity. A good example is given by the free field. Early treatments have made large use of Fock states, i.e., photon number states which are eigenstates of the free field Hamiltonian. Although they form a perfectly valid basis for the corresponding Hilbert space, these states are poorly suited for the description of laser fields which contain a large and intrinsically uncertain number of photons. To this effect another set of states, the so-called coherent states of the radiation field, sometimes called Glauber states, have been introduced and extensively studied \[1,2\]. A single term of that representation is a good lowest-order solution to important dynamical problems associated with laser emission.

The coherent states of the radiation field have attractive properties. They are obtained from the vacuum state by a unitary shift operator, and are minimum uncertainty states, i.e., products of mean square deviations of conjugated variables are minimum in these states, e.g., \[\langle \Delta p^2 \rangle \langle \Delta q^2 \rangle = \hbar^2 /4\]. Though not orthogonal, they obey a completeness relation, and hence form a good set of basis states. In fact the overcompleteness of coherent states allows one to expand many important field operators as a single integral over diagonal projectors on these states. Finally, these states correspond to the field radiated by classical currents, i.e., currents produced by moving charges for which the field reaction is neglected. In this sense these states provide a quantum description of classical fields.

One purpose of the present paper is to show that states with completely analogous properties can be defined for the free atom assembly; in fact to each property of the atomic coherent states there exists a corresponding property of the field coherent states. This duality, far from being accidental, will be shown to be deeply rooted, and related to the contraction of the rotation group, describing motions on a sphere, onto the translation group, describing motions on a plane.

For a single two-level system, that of atom \(n\), the ground-state ket will be labelled \(|\psi_1^n\rangle\) and the upper-state ket \(|\psi_2^n\rangle\). Any operator acting on this system can be expanded in the set of Pauli matrices \(\sigma_x^n, \sigma_y^n, \sigma_z^n\), plus the identity matrix \(I^n\), associated with this particular atom. The two-level system is thus identical to a spin 1/2 system for which spin-up and spin-down operators are defined by

\[
|\psi_\pm^m\rangle = \frac{1}{\sqrt{2}} (|\psi_1^m\rangle \pm i|\psi_2^m\rangle)
\]

(1.1)

For recollection, the commutation rules of these operators are

\[
[\sigma_z, \sigma_\pm] = \pm 2\sigma_\pm, \quad [\sigma_\pm, \sigma_-] = \sigma_+,
\]

(1.2)

The states \(|\psi_1^n\rangle\) and \(|\psi_2^n\rangle\) are eigenstates of \(\sigma_z^n\). Such a choice of basis is convenient but by no means unique. Any other linear combination

\[
|\psi'_1^n\rangle = \frac{1}{\sqrt{2}} \sum_{j=1}^{2} U_{ij} |\psi_j^n\rangle (i = 1, 2)
\]

(1.3)

may be chosen which preserves orthogonality and normalization. The most general transformations \(U_{ij}\) with these properties are the collection of \(2 \times 2\) unitary matrices, which form the group \(U(2)\). The subgroup of transformations with determinant +1 forms the group \(SU(2)\), familiar from angular momentum analysis. \(U(2)\) and \(SU(2)\) differ by a trivial phase factor.

Turning to the assembly of \(N\) atoms, the corresponding Hilbert space is spanned by the set of \(2^N\) product states

\[
|\psi_{112...N}\rangle = \prod_{n=1}^{N} |\psi_n^1\rangle \quad (i_n = 1, 2)
\]

(1.4)

Collective angular momentum operators are defined by

\[
J_\mu = \frac{1}{2} \sum_n \sigma_\mu^n \quad (\mu = x, y, z)
\]

(1.5a)

\[
J_\pm = \sum_n \sigma_\pm^n
\]

(1.5b)

\[
J^2 = J_x^2 + J_y^2 + J_z^2
\]

(1.5c)

For the moment the effect of the different spatial positions of atoms \(1, 2, \ldots, N\) is ignored. Following the historical development of quantum mechanics
one could choose as another suitable basis, in place of (1.4), the set of eigenstates of the energy operator $J_z$. In this case, symmetry requirements usually indicate an appropriate complete set of commuting observables to which $J_z$ belongs and whose simultaneous eigenstates form the basis. In analogy to angular momentum eigenstates these orthonormal states will be labelled

\[
\begin{pmatrix}
J \\
\lambda \\
M \\
i
\end{pmatrix}
\]

where $J(J+1)$ and $M$ are the eigenvalues of $J^2$ and $J_z$ respectively. The quantum numbers $\lambda$ and $i$ are those additional eigenvalues which are required to provide a complete set of labels. They are related to the permutation properties of the free-atom Hamiltonian as explained in a longer paper [3].

The energy eigenstates (1.6) have been used in the study of superradiance [4, 5] and will be called Dicke states. They will be shown to have a close relationship to the Fock states of the free-field problem.

Another natural way to describe the $N$ atoms is through the overcomplete set of product states

\[
|\psi_{a_1, b_1; \ldots ; a_N, b_N}\rangle = \prod_{n=1}^{N} (a_n|Y_1\rangle + b_n|Y_2\rangle)
\]

(1.7)

with $|a_n|^2 + |b_n|^2 = 1$. These states display no correlation between different atoms. For any normalized state $|Y\rangle$ of the $N$ atom assembly a degree of correlation can be defined in the following manner: one forms the overlap integral $|\langle Y |\psi_{a_1, b_1; \ldots ; a_N, b_N}\rangle|^2$ and maximizes the result with respect to the set $(a_1, b_1), \ldots, (a_N, b_N)$. The complement to one of this maximized overlap integral is defined as the degree of atomic correlation of the state $|Y\rangle$.

It can easily be seen that all states of the form (1.7) have zero correlation, whereas the Dicke states (1.6) of maximum $J (J = N/2)$ and small $M (M \approx 0)$ have a correlation which approaches unity for large $N$ values.

Another set of overcomplete states can be obtained by rotating the Dicke states $|J, -J; \lambda, i\rangle$ through an angle $|\theta, \phi\rangle$ in angular momentum space. These states, which can be labelled

\[
\begin{pmatrix}
J \\
\lambda \\
M \\
i
\end{pmatrix}
\]

(1.8)

are the atomic coherent states. They will be named Bloch states in view of their resemblance with the spin states common in nuclear induction problems [6]. Only for $J = N/2$ are these states a subset of (1.7).

The remainder of the paper is divided as follows: Section II deals with the set of equations describing the atoms-field interaction. There the self-consistent approximation to the quantum equations of motion is shown to be equivalent to a semiclassical approach. The most natural representations for this description are those whose basis vectors are coherent states, both for field and atoms. In Sec. III we recall the main properties of the coherent states for a single-field mode. Section IV describes the symmetry properties of the atomic states. The atomic coherent states representation is fully described and compared with the Dicke representation. Both in Secs. III and IV we show the virtues of the coherent state representation in dealing with statistical averages. For sake of brevity the arguments will be presented in a half-heuristic way, leaving more rigorous proofs to a longer paper [3]. Furthermore, most considerations here are limited to the interaction of a single-field mode with a collection of atoms corresponding to a single member of the set $(\lambda, i)$. The notation in Sec. IV is therefore simplified, $|J, M\rangle$, or $|M\rangle$, replacing (1.6) and $|J, \theta, \phi\rangle$, or $|\theta, \phi\rangle$, replacing (1.8). Section V explains the group contraction procedure which allows one to derive all the properties of the field states from the corresponding properties of the atomic states in Sec. IV. Appendix I gives a disentangling theorem for exponential angular momentum operators, and some properties which result, such as formulas for the coupling of rotations. These disentangling properties should find great use in many other fields of physics where rotations are considered and expectation values have to be calculated. Finally Appendix II shows an example of the application of the disentangling theorem to the calculation of a generating function for expectation values of any product of angular momentum operators in Bloch states.

II. FIELD-ATOMS INTERACTION

The interaction between the transverse electromagnetic field confined in a cavity of volume $V$ and a set of two-level atoms uniformly distributed within the cavity is described by the following Hamiltonian (in frequency units)

\[
\mathcal{H} = \sum_{k} \omega_k a_k^+ a_k + \frac{\omega_0}{2} \sum_{i=1}^{N} c_{i}^+ c_{i} + \sum_{k, i} g_k [a_k c_i^+ e^{i\vec{r}_i \cdot \vec{r}_k} + a_k^+ c_i e^{-i\vec{r}_i \cdot \vec{r}_k}] 
\]

(2.1)

where $a_k^+$, $a_k$ are the Bose operators describing $k$, the field mode; $c_i$ and $c_i^+$ are the Pauli operators for the $i$-th atom at position $\vec{r}_i$; and $g_k$ is the following coupling constant [in mks units]
quantities we would need knowledge of ternary products, and so on.

The self-consistent field approximation (SCFA) consists in assuming that binary products factor out as, e.g.,

\[ \langle a J_+ \rangle \propto \langle a \rangle \langle J_+ \rangle \]

This amounts to introducing "fluctuation" operators with zero expectation value such as

\[ \delta a = a - \langle a \rangle \]

and assuming that their correlations \( \langle \delta a \delta J_+ \rangle \) are zero. In this approximation, Eqs. (2.5) reduce to the vector equation for the atomic vector operator

\[ \langle \dot{J} \rangle = \vec{\Omega} \times \langle \vec{J} \rangle, \quad \vec{\Omega} = (2g\langle a \rangle, 0, w_0) \]  

(2.7)

(we have assumed for simplicity perfect resonance \( w = w_0 \)), and to the coupled equation

\[ \langle \dot{a} \rangle = -iw\langle a \rangle - ig\langle J_- \rangle \]

(2.8)

for the field. Equations (2.7) and (2.8) form a closed nonlinear set of equations.

For an assigned source field \( \langle a \rangle \), Eq. (2.7) describes the well-known Rabi-Bloch precession [6], responsible for displacing an atomic system from the ground state to a rotated state as (1.8). In the particular case we are considering ("point laser"), the rotated states (1.8) coincide with the uncorrelated states (1.7) as shown later in Sec. IV. Thus the Bloch states are the solution to the semiclassical problem of an atomic system illuminated by a classical field.

Vice versa, Eq. (2.8) describes the evolution of the field radiated by an assigned atomic current, proportional to \( \langle J_- \rangle \). This semiclassical problem is also well known in connection with the so-called infrared catastrophe [8,1,2].

The states reached by a quantum field generated from the ground state by a classical current are those states that will be considered in detail in Sec. III, and that we have named Glauber states.

By joining the two semiclassical considerations, it results that a suitable (even though not necessarily unique) choice for Schrödinger states of a field–atom system starting from a "vacuum" (no photons in the field, atoms all excited) at \( t=0 \) are the product states of Glauber and Bloch states as suggested by Eqs. (2.7) and (2.8). This heuristic consideration shows the kinship between Glauber and Bloch states. Such a kinship will appear more dramatically in Secs. III and IV.
III. DESCRIPTION OF THE FREE FIELD

A. The Harmonic Oscillator States

In order to point out with maximum clarity the analogies between the free-field description and the free-atom description we start by listing here, in simple terms, the properties of the single harmonic oscillator. The equation numbering in this section and in Sec. IV is done in parallel.

The single harmonic oscillator is described by its canonically conjugated coordinates \( q, p \) with the commutation relation

\[
[q, p] = i\hbar
\]

(3.1)

One forms the usual lowering and raising operators

\[
a = (2\hbar m)^{-1/2} (\hbar \omega q + ip)
\]

(3.2a)

\[
a^\dagger = (2\hbar m)^{-1/2} (\hbar \omega q - ip)
\]

(3.2b)

Where \( \hbar \omega m > 0 \) is characteristic of the oscillator. These operators satisfy

\[
[a, a^\dagger] = 1
\]

(3.3a)

from which one obtains

\[
[a, a^\dagger a] = a
\]

(3.3b)

\[
[a^\dagger, a^\dagger a] = -a^\dagger
\]

(3.3c)

The harmonic oscillator states, or Fock states, are the eigenstates of

\[
N = a^\dagger a
\]

(3.4)

and are given by [9]

\[
|n\rangle = \frac{(a^\dagger)^n}{\sqrt{n!}} |0\rangle \quad (n = 0, 1, 2, \ldots)
\]

(3.5)

with eigenvalue \( n \). The vacuum state \( |0\rangle \) is the harmonic oscillator ground state defined by

\[
a |0\rangle = 0
\]

(3.6)

B. Coherent States of the Field

Let us consider the translation operator which produces a shift \( \xi \) in \( q \) and \( \eta \) in \( p \):

\[
T_\alpha = \exp[-(i/\hbar)(\xi p - \eta q)] = \exp(\alpha a^\dagger - \alpha^* a)
\]

(3.7)

where

\[
\alpha = (2\hbar m)^{-1/2} (\hbar \omega \xi + i\eta)
\]

(3.7b)

A coherent state \( |\alpha\rangle \) is obtained by translation of the ground state \([1, 2]\)

\[
|\alpha\rangle = T_\alpha |0\rangle
\]

(3.8)

We shall name these states Glauber states, as they have been used extensively by Glauber in quantum optics [2]. Since

\[
T_\alpha a T_\alpha^{-1} = a - \alpha
\]

(3.9)

the state \( |\alpha\rangle \) satisfies the eigenvalue equation

\[
(a - \alpha)|\alpha\rangle = 0
\]

(3.10)

Using the Baker-Campbell-Hausdorff theorem [10] or Feynman disentangling techniques [11] the translation operator can be written in the following forms:

\[
T_\alpha = e^{\alpha |\alpha|^2/2} e^{-\alpha^* a} e a^\dagger e^{-\alpha |\alpha|^2/2} e a^\dagger e^{-\alpha^* a}
\]

(3.11)

The second of these forms, which is known as the normally ordered form, gives immediately the expansion of \( |\alpha\rangle \) in terms of Fock states

\[
|\alpha\rangle = T_\alpha |0\rangle = e^{-|\alpha|^2/2} e a^\dagger |0\rangle
\]

(3.12)

from which, expanding the exponential and using (3.5), one obtains

\[
\langle n |\alpha\rangle = e^{-|\alpha|^2/2} \frac{\alpha^n}{\sqrt{n!}}
\]

(3.13)

The scalar product of Glauber states can be obtained either from (3.12), using the disentangling theorem (3.11), or from (3.13), using the completeness property of Fock states, \( \Sigma \langle n | n \rangle = 1 \). One gets
\[ \langle \alpha | \beta \rangle = \exp(-1/2)(|\alpha|^2 - 2\alpha^* \beta + |\beta|^2) \] (3.14a)

from which one obtains

\[ |\langle \alpha | \beta \rangle|^2 = \exp(-|\alpha - \beta|^2) \] (3.14b)

The coherent states are minimum uncertainty packets. For three observables \( A, B, C \), which obey a commutation relation \( [A, B] = i\hbar \), it is easy to show [9] that \( \langle A^2 \rangle \langle B^2 \rangle \geq \langle C \rangle^2/4 \). In particular, with \( A = q-\xi \), \( B = p-\eta \) and \( C = \hbar \) one has

\[ \langle (q - \xi)^2 \rangle \langle (p - \eta)^2 \rangle \geq \hbar^2/4 \] (3.15)

for any state. It is easy to show [2] that the equality sign holds for the coherent state \( |\alpha\rangle \), where \( \alpha \) is related to \( \xi \) and \( \eta \) by (3.7b). This establishes the minimum uncertainty property.

C. The Coherent States as a Basis

We now consider the completeness properties of the coherent states. Using (3.13), and the completeness of Fock states \( \sum_n |n\rangle \langle n| = 1 \), one obtains straightforwardly

\[ \int \frac{d^2 \alpha}{\pi} |\alpha\rangle \langle \alpha| = 1 \] (3.16)

The expansion of an arbitrary state in Glauber states follows

\[ |c\rangle = \sum_n C_n |n\rangle = \int \frac{d^2 \alpha}{\pi} \sum_n C_n |\alpha\rangle \langle \alpha| n\rangle = \int \frac{d^2 \alpha}{\pi} \exp(-1/2)|\alpha|^2 f(\alpha^*) |\alpha\rangle \] (3.17a)

where

\[ f(\alpha^*) = \sum_n C_n \frac{\alpha^* n}{\sqrt{n!}} = e^{-|\alpha|^2/2} \langle \alpha | c \rangle \] (3.17b)

Using (3.5) one sees that \( |c\rangle \) can also be written as

\[ |c\rangle = f(\alpha^+) |0\rangle \] (3.18)

where \( f(\alpha^+) \) is defined by its expansion (3.17b). The scalar product of any two states \( |c'\rangle \) and \( |c\rangle \) is obtained from (3.16) and (3.17b)

\[ \langle c' | c \rangle = \int \frac{d^2 \alpha}{\pi} \langle c' | \alpha \rangle \langle \alpha | c \rangle \]

\[ = \int \frac{d^2 \alpha}{\pi} e^{-|\alpha|^2} [f(\alpha^*)^* f(\alpha)] \] (3.19)

In view of the completeness relations, operators \( F \) acting on this Hilbert space can be expanded as

\[ F = \sum_{m,n} |m\rangle \langle m| F |n\rangle \langle n| \] (3.20a)

or

\[ F = \int \frac{d^2 \alpha}{\pi} \int \frac{d^2 \beta}{\pi} |\beta\rangle \langle \beta| F |\alpha\rangle \langle \alpha| \] (3.20b)

Due to the overcompleteness of the \( |\alpha\rangle \) states, the expansion (3.20b) is in general not unique. This expansion is especially useful if it can be written in the diagonal form

\[ F = \int d^2 \alpha f(\alpha) |\alpha\rangle \langle \alpha| \] (3.20c)

This will be further discussed for the case of the density matrix.

D. Statistical Operator for the Field

Up to now we have considered pure quantum states. Since a field in thermal equilibrium with matter at ordinary temperatures is essentially in the ground state \( (\omega \gg \hbar k_B T), \) this is an adequate description for any field obtained from thermal equilibrium in response to a classical current. However, the field radiated by an incoherently pumped medium is a statistical mixture described by a statistical operator \( \rho \), which we assume normalized to unity,

\[ \text{Tr} \rho = 1 \] (3.21)

With the help of this operator, the statistical average of any observable \( F(a, a^\dagger) \) is obtained as

\[ \langle F \rangle = \text{Tr} \rho F \] (3.22)
Of particular interest are statistical ensembles described by a statistical operator which is diagonal in the Glauber representation [2]

\[ \rho = \int P(\alpha) |\alpha\rangle \langle \alpha| \, d^2\alpha \]  

(3.23)

where the normalization (3.21) requires

\[ \int P(\alpha) \, d^2\alpha = 1 \]  

(3.24)

The statistical average of an observable F is then given by an average over the diagonal elements \( \langle \alpha | F | \alpha \rangle \):

\[ \langle F \rangle = \int P(\alpha) \langle \alpha | F | \alpha \rangle \, d^2\alpha \]  

(3.25)

The weight function \( P(\alpha) \) has thus the properties of a distribution function in \( \alpha \)-space, except that it is not necessarily positive.

Let us define a set of operators \( \hat{X}(\lambda) \) such that their expectation values for coherent states

\[ \langle \alpha | \hat{X}(\lambda) | \alpha \rangle = b^2(\lambda) \]  

(3.26)

form a basis in the function space of functions of \( \alpha \). If the statistical ensemble has a diagonal representation (3.23), then the statistical averages of the operators form a kind of "characteristic function" of \( P(\alpha) \):

\[ X(\lambda) \equiv \langle \hat{X}(\lambda) \rangle = \int d^2\alpha \, P(\alpha) b^2(\lambda) \]  

(3.27)

The weight function \( P(\alpha) \) can be expressed in terms of \( X(\lambda) \) with the help of the reciprocal basis \( b^*_{\lambda}(\alpha) \),

\[ P(\alpha) = \int d^2\lambda \, X(\lambda) \, b^*_{\lambda}(\alpha) \]  

(3.28)

A convenient basis is the Fourier basis

\[ b^{\alpha}(\lambda) = \exp(\lambda a^* - \lambda^* a) \]  

(3.29a)

\[ b^{\lambda}(\alpha) = (2\pi)^{-2} \exp(-\lambda a^* + \lambda^* a) \]  

(3.29b)

which is generated by the normally ordered operators

\[ \hat{X}_N(\lambda) = \exp(\lambda a^\dagger) \exp(-\lambda^* a) \]  

(3.29c)

The question of the existence of the \( P \) representation is a complicated one.

[2,12]. Using the Fourier basis (3.29) it can be shown however that the mere existence of the inverse transformation (3.28) guarantees that the resulting function \( P(\alpha) \) can be used to calculate the statistical average of any moment \( \langle a^{m*} a^n \rangle \) as if \( P(\alpha) \) was the weight function defined in (3.23). This is due to the fact that the characteristic function \( X_N(\lambda) \) plays the role of a generating function for \( \langle a^{m*} a^n \rangle \):

\[ \langle a^{m*} a^n \rangle = \left( \frac{\lambda}{\delta \lambda} \right)^n \left( \frac{-\lambda^*}{\delta \lambda^*} \right)^m X_N(\lambda) |_{\lambda=0} \]

From which, by derivation of (3.27), one obtains

\[ \langle a^{m*} a^n \rangle = \int d^2\alpha \, P(\alpha) \langle \alpha | a^{m*} a^n | \alpha \rangle \]

which is a particular case of (3.25) and proves the above statement. One could moreover introduce, in addition to (3.29c), symmetrically ordered \( \hat{X}_S(\lambda) \), and antinormally ordered \( \hat{X}_A(\lambda) \), exponential operators [2,10]. The Fourier transform of their statistical averages are the Wigner distribution, and the matrix element \( \langle \alpha | e^{\alpha \hat{a}^\dagger} e^{\alpha^* \hat{a}} | \alpha \rangle \), respectively. We will not develop these aspects further as the corresponding expressions for atomic coherent states are rather involved, and of no clear use as yet.

IV. DESCRIPTION OF THE FREE ATOMS

A. The Angular Momentum States

As shown in Sec. 1, angular momentum operators can be defined which act on the \( N \)-atom Hilbert space. In particular we can consider a subspace of degenerate eigenstates of \( J^2 \) with eigenvalues \( J(J+1) \). Since \( J^2 \) commutes with \( J_x, J_y, J_z \), these operators only connect states within the same subspace. In general, \( J^2 \) and \( J_z \) do not form a complete set of commuting observables. As explained in Ref. 3, such a complete set is formed by adding to \( J_z \) and \( J_x \) some operators of the permutation group of \( N \) objects \( \pi_N \). These operators play with respect to \( \pi_N \) the same role that \( J^2 \) and \( J_z \) have with respect to the three-dimensional rotation group. We shall assume that the subspace considered here has also been made invariant under these permutation operations, but for simplicity we shall omit for the time being to indicate this in the labelling of the states. The subspace we are dealing with is identical to a constant angular momentum Hilbert space. The Dicke states, which are the analog of the Fock states (3.5), and the Bloch states, which correspond to the Glauber state (3.8), are most easily
defined within such a subspace. The equation numbering is in parallel with that of Sec. III. From the angular momentum operators $J_x$ and $J_y$, which satisfy the commutation relation

$$[J_x, J_y] = iJ_z$$

(4.1)

the lowering and raising operators are formed

$$J_- = J_x - J_y$$

(4.2a)

$$J_+ = J_x + J_y$$

(4.2b)

which obey

$$[J_-, J_+] = 2J_z$$

(4.3a)

$$[J_-, J_z] = J_-$$

(4.3b)

$$[J_+, J_z] = -J_+$$

(4.3c)

The Dicke states, which are simply the usual angular momentum states, are defined as the eigenstates of

$$J_z = \frac{1}{2} (J_+ J_- - J_- J_+)$$

(4.4)

They are given by [9,13]

$$|M\rangle = \frac{1}{(M + J)!} \left( \frac{2J}{M + J} \right)^{1/2} J_{M+J} |\bar{J}\rangle$$

(4.5)

(M = \bar{J}, \bar{J} + 1, \ldots, J)

with eigenvalue $M$. They span the space of angular momentum quantum number $J$. The ground state $|\bar{J}\rangle$ is defined by

$$J_- |\bar{J}\rangle = 0$$

(4.6)

B. Coherent Atomic States

Let us consider the rotation operator which produces a rotation through an angle $\theta$ about an axis $\hat{n} = (\sin\varphi, \cos\varphi, 0)$:

$$R_{\theta, \varphi} = e^{-i\varphi J_n} = \exp[-i(\varphi J_x \sin\varphi - J_y \cos\varphi)]$$

$$= \exp(\varphi J_+ - \varphi J_-)$$

(4.7a)

where

$$\varphi = \frac{\theta}{2} e^{-i\varphi}$$

(4.7b)

A coherent atomic state, or Bloch state, $|\theta, \varphi\rangle$ is obtained by rotation of the ground state $|\bar{J}\rangle$:

$$|\theta, \varphi\rangle = R_{\theta, \varphi} |\bar{J}\rangle$$

(4.8)

Furthermore

$$R_{\theta, \varphi} J_n R_{\theta, \varphi}^{-1} = J_n$$

$$R_{\theta, \varphi} J_k R_{\theta, \varphi}^{-1} = J_k \cos\theta + J_z \sin\theta$$

$$R_{\theta, \varphi} J_z R_{\theta, \varphi}^{-1} = -J_k \cos\theta + J_z \sin\theta$$

where

$$J_n = J_x \sin\varphi - J_y \cos\varphi$$

$$J_k = J_x \cos\varphi + J_y \sin\varphi$$

which gives

$$J_+ = (J_k - iJ_n) e^{i\varphi}$$

$$J_- = (J_k + iJ_n) e^{-i\varphi}$$

Using these relations one obtains

$$R_{\theta, \varphi} J_+ R_{\theta, \varphi}^{-1} = e^{-i\varphi} [J_- e^{i\varphi} \cos^2 (\theta/2) - J_+ e^{-i\varphi} \sin^2 (\theta/2) + J_z \sin\theta]$$

(4.9a)

and similar relations for $J_-$ and $J_z$:

$$R_{\theta, \varphi} J_- R_{\theta, \varphi}^{-1} = e^{-i\varphi} [J_+ e^{-i\varphi} \cos^2 (\theta/2) - J_- e^{i\varphi} \sin^2 (\theta/2) + J_z \sin\theta]$$

(4.9b)
From (4.9a), and definition (4.8), one obtains the eigenvalue equation

\[ [J_e^{-1} e^{i\tau} \cos^2(\theta/2) - J_+ e^{-i\tau} \sin^2(\theta/2) + J_z \sin \theta] \mid \theta, \varphi \rangle = 0 \]  

(4.10a)

This equation, together with

\[ J^2 \mid \theta, \varphi \rangle = J(J+1) \mid \theta, \varphi \rangle \]  

(4.10b)

specifies uniquely the Bloch state \( \mid \theta, \varphi \rangle \). Note that the harmonic oscillator analog of (4.10b) would have been the trivial relation \( (\tau^* - \tau)(\tau - \tau^*) \mid \alpha \rangle = 0 \).

Other forms of the eigenvalue equation can be obtained using the relation

\[ R \mid \theta, \varphi \rangle = \mid \theta, \varphi \rangle \]  

and (4.9c). The resulting equation can be combined with (4.10a) to eliminate one of the operators \( J_z, J_+ \), or \( J_- \), giving

\[ [J_e^{-1} e^{i\tau} \cos^2(\theta/2) + J_+ e^{-i\tau} \sin^2(\theta/2)] \mid \theta, \varphi \rangle = J \sin \theta \mid \theta, \varphi \rangle \]  

(4.10c)

\[ [J_e^{-1} e^{i\tau} \cos(\theta/2) + J_z \sin(\theta/2)] \mid \theta, \varphi \rangle = J \sin(\theta/2) \mid \theta, \varphi \rangle \]  

(4.10d)

\[ [J_+ e^{-i\tau} \sin(\theta/2) - J_z \cos(\theta/2)] \mid \theta, \varphi \rangle = J \cos(\theta/2) \mid \theta, \varphi \rangle \]  

(4.10e)

These additional relations are not independent of (4.10a) and (4.10b). One notes that these eigenvalue equations are more complicated than their counterpart (3.10). In particular they involve at least two of the three operators \( J_-, J_+, J_z \). This feature is required by the more complicated commutation relation (4.1) which applies here.

Using the disentangling theorem for angular momentum operators (Appendix I), the rotation \( R_{\theta, \varphi} \) given by (4.7a) becomes

\[ R_{\theta, \varphi} = \exp(-\tau J_-) \exp[-\ln(1 + |\tau|^2) J_z] \exp(\tau J_+) \]  

(4.11a)

\[ = \exp(\tau J_+) \exp[\ln(1 + |\tau|^2) J_z] \exp(-\tau J_-) \]  

(4.11a)

where

\[ \tau = e^{-i\varphi} \tan(\theta/2) \]  

(4.11b)

Let us point out that these expressions are singular for \( \theta = \pi \), i.e., for the uppermost state. We may have to exclude from some of the following considerations the states contained within an infinitesimally small circle around \( \theta = \pi \). The validity of expressions such as (4.13) for \( \theta = \pi \) is usually not affected and can be checked directly. The last form of (4.11a), which we call the normalized form, gives immediately the expansion of \( \mid \theta, \varphi \rangle \) in terms of Dicke states:

\[ \mid \theta, \varphi \rangle = R_{\theta, \varphi} \mid J \rangle = \frac{1}{1 + |\tau|^2} \frac{\sin J^+ M^+ J}{\sin(\theta/2) \sin(\theta/2)} \mid J \rangle \]  

(4.12)

from which, expanding the exponential and using (4.5) one obtains

\[ \langle M \mid \theta, \varphi \rangle = \frac{2 J^M}{{J^M + 1} \/ \frac{1 + |\tau|^2}{2} \sin J^+ M^+ J / \sin(\theta/2) \sin(\theta/2) e^{-i\tau J^M \varphi} \]  

(4.13)

Since the Dicke states form a basis for a well-known irreducible representation of the rotation group, these results could have been derived using the appropriate Wigner \( \Theta(J) \) matrix [13]. The same remark applies to Eqs. (3.12) and (3.13): these could have been obtained without using the Baker-Campbell-Hausdorff formula, from the transformation properties of an irreducible representation of the group of operations \( T_c \).

The overlap of two Bloch states is obtained either from (4.12), using the disentangling theorem for exponential angular momentum operators, or from (4.13), using the completeness property of Dicke states \( \Sigma_M \langle M \mid M \rangle = 1 \). One obtains

\[ \langle \theta, \varphi \mid \theta', \varphi' \rangle = \frac{1}{1 + |\tau|^2} \frac{\sin J^+ M^+ J}{\sin(\theta/2) \sin(\theta/2)} \]  

\[ = e^{iJ(\varphi - \varphi')} \left( \cos \frac{\theta - \theta'}{2} \cos \frac{\omega - \omega'}{2} - i \cos \frac{\theta + \theta'}{2} \sin \frac{\omega - \omega'}{2} \right) \]  

(4.14a)

from which one obtains

\[ |\langle \theta, \varphi \mid \theta', \varphi' \rangle|^2 = \cos^4(J/2) \]  

(4.14b)

where \( \tau \) is given by (4.11b), \( \tau' \) is given by the same equation written with the primed quantities, and \( \theta \) is the angle between the \( (\theta, \varphi) \) and \( (\theta', \varphi') \) directions, as given by
\[ \cos \theta = \cos \theta' + \sin \theta' \cos(\varphi - \varphi') \]

The Bloch states form minimum uncertainty packets. The uncertainty relation can be defined in terms of the set of rotated operators 
\( (J_x', J_y', J_z') = R_{\theta} (J_x', J_y', J_z') R^{-1} \). These three observables obey a commutation relation of the type 
\[ [A, B] = i\hbar C \] with \( A = J_x', B = J_y', C = J_z' \), from which they have the uncertainty property
\[ \langle J_x'^2 \rangle \langle J_y'^2 \rangle \geq \frac{1}{4} \langle J_z'^2 \rangle^2 \] (4.15)

for any states. It is easy to show that the equality sign holds for the Bloch state \( |\theta, \varphi \rangle \), which is therefore a minimum uncertainty state.

C. The Bloch States as a Basis

Let us now consider the completeness properties of the Bloch states. Using (4.13), and the completeness of Dicke states \( \sum_M |M\rangle \langle M| = 1 \), one obtains
\[ (2J+1) \frac{d\Omega}{4\pi} |\theta, \varphi \rangle \langle \theta, \varphi | \]
\[ = (2J+1) \frac{d\Omega}{4\pi} \sum_{M,M'} \left( 2J \frac{1}{2} \frac{1}{2} \frac{1}{2} \sin \frac{\theta}{2} \right) (M'-M) \varphi \]
\[ \times (\cos \frac{\theta}{2})^{2J-M-M'} (\sin \frac{\theta}{2})^{2J+M+M'} |M\rangle \langle M'| \]
\[ = (2J+1) \int_0^\pi d\theta \frac{\sin \theta}{2} \sum_M \left( 2J \frac{1}{2} \frac{1}{2} \frac{1}{2} \sin \frac{\theta}{2} \right) (M+J) (M+J') \varphi \]
\[ \times (\cos \frac{\theta}{2})^{2J-2M} (\sin \frac{\theta}{2})^{2J+2M} |M\rangle \langle M| \]
\[ = \sum_M |M\rangle \langle M| = 1 \] (4.16)

The expansion of an arbitrary state in Bloch states follows:
\[ |c\rangle = \sum_M C_M |M\rangle = (2J+1) \frac{d\Omega}{4\pi} \sum_M C_M |\theta, \varphi \rangle \langle \theta, \varphi | |M\rangle \]
\[ = (2J+1) \frac{d\Omega}{4\pi} \frac{f(\tau^*)}{(1 + |\tau|^2)^J} |\theta, \varphi \rangle \] (4.17a)

where
\[ f(\tau^*) = \sum_M C_M \left( 2J \frac{1}{2} \frac{1}{2} \frac{1}{2} \right)^{1/2} (\tau^*)^J \]
\[ = (1 + |\tau|^2)^J \langle \theta, \varphi | c \rangle \] (4.17b)

Using (4.5) one sees that \( |c\rangle \) can also be written as
\[ |c\rangle = \frac{1}{\sqrt{2J+1 - J}} J \langle J^+ \rangle \]
(4.18)

The amplitude function \( f(\tau^*) \) is, by its definition (4.17b), a polynomial of degree \( 2J \). However any function which has a Maclaurin expansion can be taken as a suitable amplitude function in (4.17a) or (4.18). Indeed the powers of \( \tau^* \) higher than \( 2J \) gives zero contribution in (4.17a) and (4.18).

The coefficients \( C_M \) are then obtained from the first \( (2J+1) \) terms of the Maclaurin series, using (4.17b).

The scalar product of two states characterized by their amplitude function is, from (4.16) and (4.17b)
\[ \langle c' | c \rangle = \]
\[ = (2J+1) \int d\Omega \frac{\sin \theta}{2} \sum_M \left( 2J \frac{1}{2} \frac{1}{2} \frac{1}{2} \sin \frac{\theta}{2} \right) (M+J) (M+J') \varphi \]
\[ \times (\cos \frac{\theta}{2})^{2J-2M} (\sin \frac{\theta}{2})^{2J+2M} |M\rangle \langle M| \]
\[ = \sum_M |M\rangle \langle M| = 1 \] (4.17c)

Since (4.17b) was used to derive this equation, its validity is restricted to amplitude functions which are polynomials of degree \( 2J \).

In view of the completeness relations, operators \( G \) acting on this Hilbert space can be expanded as
\[ G = \sum_{M,M'} |M\rangle \langle M| G |M\rangle \langle M'| \]
(4.20a)
or
\[ G = \frac{(2J+1)^2}{(4\pi)^2} \int d\Omega d\Omega' |\theta, \varphi \rangle \langle \theta, \varphi| G |\theta', \varphi' \rangle \langle \theta', \varphi'| \]
(4.20b)

However, \( G \) is completely defined by the \( (2J+1)^2 \) matrix elements \( \langle M| G |M'\rangle \). It results that, except for pathological cases, an operator can always be written in the diagonal form
\[ G = \int d\Omega g(\theta, \varphi) |\theta, \varphi \rangle \langle \theta, \varphi| \]
(4.20c)

where \( g(\theta, \varphi) \) is given by a series expansion.
statistical averages of the operators \( \hat{X}_\lambda \) form a set of "characteristic coefficients" of \( P(\theta, \varphi) \):

\[
X_\lambda = \text{Tr} \rho \hat{X}_\lambda = \int d\Omega P(\theta, \varphi) b_\lambda^*(\theta, \varphi)
\]

(4.27)

The weight function can be expressed as a series with the help of the reciprocal basis \( b_\lambda(\theta, \varphi) \),

\[
P(\theta, \varphi) = \sum_\lambda X_\lambda b_\lambda(\theta, \varphi)
\]

(4.28)

A convenient basis is given by the spherical harmonics

\[
b_\lambda(\theta, \varphi) = Y_\lambda^m(\theta, \varphi)
\]

(4.29a)

\[
\bar{b}_\lambda(\theta, \varphi) = Y_\lambda^{-m}(\theta, \varphi)
\]

(4.29b)

which are generated by the spherical harmonic operators [14]

\[
\hat{X}_\lambda = Y_\lambda^m(\hat{J})
\]

(4.29c)

The already mentioned fact that a diagonal representation always exists in the atomic case also corresponds to the fact that for a given \( J \) only the \((2J + 1)^2\) operators \( Y_\lambda^m \) with \( \lambda \leq 2J \) are different from zero. The finite dimensionality of the basis is required, since \( \rho \) is completely determined by its \((2J + 1)^2\) matrix elements \( \langle M|\rho|M' \rangle \) in the Dicke representation.

Other differences with the field case should also be noted. First, the spherical harmonic operators are usually written in a fully symmetrized form, whereas the operators (3.29c) are normally ordered. This is only a formal difficulty as it should be possible to write normally ordered and anti-normally ordered multiple operators with properties similar to the \( Y_\lambda^m(\hat{J}) \).

A second, and more fundamental, difference is that the expectation values \( X_\lambda \) are not generating functions for products of the type \( \langle \hat{J}_x \hat{J}_y \hat{J}_z \rangle \), in view of the discreteness of the set. This does not cause much difficulty, as generating functions can be derived from exponential operators (Appendix II) whose expectation values can be calculated with the help of the disentangling theorem (Appendix I). It is tempting to use for the \( X_\lambda \)'s of Eq. (4.29c) these exponential operators themselves. Though the parallel with the field case then seems more transparent, the use of the discrete set \( \hat{X}_\lambda \) may be of more fundamental significance as it takes into account symmetry properties of the states.

A final comment should be made about the difficulty of dealing with
creation and annihilation operations in a finite Hilbert space. The existence of two terminal states, \(|J\rangle\) and \(|-J\rangle\), requires the presence of a third operator with the properties of \(J_{z,p}^{-1}\), and prevents the writing of an eigenvalue equation in terms of one compound operator alone. For instance, the comparison of (4.18) and (3.18) suggests that \(J_{z,p}(J + 1 - J_{z,p}^{-1})\) could be a "good" annihilation operator. Using (4.13) one finds immediately

\[
J_{z,p}(J + 1 - J_{z,p}^{-1}) \begin{bmatrix} 0, \varphi \end{bmatrix} = \tau \begin{bmatrix} 0, \varphi \end{bmatrix} - \tau \sin \frac{2\varphi}{\varphi} e^{-2i\Delta \varphi} \begin{bmatrix} J \end{bmatrix}
\]

which for small \(\varphi\) and large \(J\) is almost an eigenvalue equation: so the application of the operator reproduces \(|0, \varphi\rangle\) except for the uppermost Dicke state. There is no doubt that a theory could be developed in terms of more complicated annihilation and creation operators of such type, but the advantages are not clear.

V. CONTRACTION, OR THE RELATION BETWEEN ATOMIC STATES AND FIELD STATES

The extreme similarity between the treatments of Secs. IV and III suggests a close connection between atomic and field states. This connection is made here through a process known as group contraction. The time evolution of a single two-level atom is governed by a \(2 \times 2\) unitary transformation matrix. The commutation relations for the generators of the group U(2) are rewritten here:

\[
\begin{align*}
[J_{z,p}, J_{z,q}] &= \pm J_{z,p} \\
[J_{z,p}, J_{z,q}] &= 2 J_{z,p} \\
[J_{z,p}, J_{p,q}] &= 0
\end{align*}
\]

(5.1)

where \(J_{z,p}\) in the third relation is essentially the identity. An arbitrary \(2 \times 2\) unitary transformation matrix is given by

\[
U(2) = \exp(i \sum_{\mu} \lambda_{\mu} J_{\mu})
\]

(5.2)

where the summation is over all four indices and the \(\lambda_{\mu}\)'s are c-number parameters which characterize the group operation.

If another set of generators \(h_{+}, h_{-}, h_{z}, h_{0}\) is related to \(J_{+}, J_{-}, J_{z}, J_{0}\) by a nonsingular transformation \(A_{\nu\mu}\)

\[
h_{\nu} = \sum_{\mu} A_{\nu\mu} J_{\mu}
\]

(5.3a)

then the group operation (5.2) may be written

\[
\exp(i \sum_{\nu} \lambda_{\nu} h_{\nu}) = \exp(ia_{\nu} h_{\nu})
\]

(5.4)

with

\[
a_{\nu} = \sum \lambda_{\mu} (A^{-1})_{\mu\nu}
\]

(5.3b)

We select the following transformation \(A\), which depends on a real parameter \(c\):

\[
\begin{bmatrix} h_{+} \\ h_{-} \\ h_{z} \\ h_{0} \end{bmatrix} = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & c & 0 & 0 \\ 0 & 0 & 1/2c^2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} J_{+} \\ J_{-} \\ J_{z} \\ J_{0} \end{bmatrix}
\]

(5.5)

It is easily verified that the \(h_{\nu}\)'s satisfy the commutation relations

\[
[h_{z}, h_{\pm}] = \pm h_{\pm}
\]

\[
[h_{+}, h_{-}] = 2c^2 h_{z} - h_{0}
\]

\[
[h, h_{0}] = 0
\]

(5.6)

In the limit \(c \to 0\) the transformation \(A\) becomes singular and \(A^{-1}\) fails to exist. Nevertheless, the commutation relations (5.6) are well defined, and in fact identical to the commutation relations (3.6) under the identification

\[
\begin{align*}
\lim_{c \to 0} h_{z} &= n = aT; & \lim_{c \to 0} h_{\pm} &= aT; & \lim_{c \to 0} h_{0} &= a \\
c -0 & & c -0 & & c -0
\end{align*}
\]

(5.7)

Although the inverse \(A^{-1}\) (5.3b) does not exist as \(c \to 0\), the parameter \(a_{\nu}\) may approach a well-defined limit if we demand all the transformations \(h_{\mu}\) to shrink ("contract") to zero in the limit \(c \to 0\), in such a way that the following ratios are well defined:

\[
\lim \frac{h_{\mu}}{c} = \lim \frac{e^{-i\theta}}{2\pi} = \alpha
\]
\[ \lim_{c \to 0} \frac{1}{c} = \lim_{\theta \to 0} e^{i \omega \theta / 2c} = -e^{i \omega \theta} \]

\[ \lim_{c \to 0} \frac{\lambda_z}{c} = \lim_{c \to 0} c \frac{\lambda_z}{c^2} = 0 \quad (5.8) \]

Within any \((2J + 1)\)-dimensional representation of the group \(U(2)\) the eigenvalue of the diagonal operator \(\lambda_z\) is

\[ h_z \left| J, M \right> = (J_z + 1/2c^2) \left| J, M \right> = \left| J, M \right> (M + 1/2c^2) \quad (5.9) \]

We demand this have a definite limit as \(c \to 0\). Physically, for both Fock and Dicke states we progress upward from the ground or vacuum state. It is convenient to demand that the (energy) eigenvalue in (5.9) be zero for the ground state \(M = -J\)

\[ \lim_{c \to 0} (-J + 1/2c^2) = 0 \quad (5.10) \]

In the limit \(c \to 0\), \(2Jc^2 = 1\), the unitary irreducible representation \(D^J[U(2)]\) goes over into the unitary irreducible representation for the contracted group with generators (5.6). In simple words, the contraction procedure amounts to letting the radius of the Bloch sphere tend to infinity as \(1/c^2\), while considering smaller and smaller rotations on the sphere. The motion on the sphere then becomes identical to the motion on the bottom tangent plane which goes over into the phase plane of the harmonic oscillator.

This procedure for contracting groups, commutation relations, and representations will now be used to show the similarity between Dicke and Fock states. We define

\[ \left| \omega, n \right> = \lim_{c \to 0} \left| J, M \right> \quad (J + M = n \text{ fixed}) \quad (5.11) \]

Then

\[ a^+ a \left| \omega, n \right> = \lim_{c \to 0} (J_z + 1/2c^2) \left| J, M \right> = \lim_{c \to 0} \left| J, M \right> [J + M + (-J + 1/2c^2)] = n \left| \omega, n \right> \quad (5.12a) \]

The computations for \(a^+\) and \(a\) are handled in an entirely analogous way:

\[ a^+ \left| \omega, n \right> = \lim_{c \to 0} h_+ \left| J, M \right> = \sqrt{n+1} \left| \omega, n+1 \right> \quad (5.12b) \]

\[ a \left| \omega, n \right> = \lim_{c \to 0} h_- \left| J, M \right> = \sqrt{n} \left| \omega, n-1 \right> \quad (5.12c) \]

These equations provide a straightforward connection between Dicke and Fock states. The operators \(h_+, h_-, h_z\) contract to \(a^+, a, a^+a\) with the proper commutation relations (3.3), and with the proper matrix elements between contracted Dicke states as shown in (5.12). The contracted Dicke states (5.11) can thus be identified with the Fock states, and we conclude that every property of Dicke and Bloch states listed in Sec. IV must contract to a corresponding property of Fock and Glauber states listed in Sec. III. The contraction procedure is summarized in Table I.

### Table I. Rules for Contraction of the Angular Momentum Algebra to the Harmonic Oscillator Algebra.

[The limit of the angular momentum quantities (1st line) for \(c \to 0\) are the corresponding harmonic oscillator quantities (2nd line).]

<table>
<thead>
<tr>
<th>Operators</th>
<th>Coordinates</th>
<th>Eigenvalues</th>
<th>Eigenstates</th>
<th>Coherent states</th>
</tr>
</thead>
<tbody>
<tr>
<td>Angular momentum</td>
<td>(cJ_+, cJ_-, J_z + \frac{1}{2c^2})</td>
<td>(\frac{3}{2c} e^{-i\theta})</td>
<td>(2c^2J, J + M)</td>
<td>(\left</td>
</tr>
<tr>
<td>Harmonic oscillator</td>
<td>(a^+, a, a^+a)</td>
<td>(a)</td>
<td>(1, n)</td>
<td>(\left</td>
</tr>
</tbody>
</table>

We demonstrate this correspondence in some particular cases:

Example 1: Just as the angular momentum eigenstates \(\left| J, -J \right>\) by \((J+M)\) successive applications of the shift-up operator \(J_+\), the Fock state \(\left| n \right>\) is obtained by \(n\) successive applications of \(a^+\). By contraction of (4.5) we get

\[ \left| \omega, n \right> = \lim_{c \to 0} \frac{(J_+)^n}{[2J n!/(2J-n)!]^{1/2}} \left| J, -J \right> \]

\[ = \lim_{c \to 0} \frac{(cJ_+)^n}{[(2Jc^2)^n n!]^{1/2}} \left| J, -J \right> \]
which is nothing but (3.5).

Example 2: Let us now contract Bloch states to Glauber states, using equation (4.12) and Table I:

\[
|\alpha\rangle = \lim_{n \to 0} |\alpha, \varphi\rangle = \lim_{n \to 0} \left( \frac{1 + |\alpha|^2}{1 + |\alpha|^2} \right)^n e^{|\alpha|^2} |\varphi\rangle \\
= \lim_{n \to 0} \left( 1 - 2 \alpha^2 (\alpha^2/2) \right)^{1/2} e^{\alpha^2} |0\rangle \\
= e^{-\alpha^2/2} e^{\alpha^2} |0\rangle
\]  

which is nothing but (3.12).

We leave it to the reader to verify that every equation of Sec. IV goes over to the corresponding equation of Sec. III under contraction. This is true in particular of the disentangling theorem (4.11) whose contracted limit is the Baker-Campbell-Hausdorff formula (3.11) [cf. Appendix I, Eq. (A1.4)]. In general all properties related to angular momentum have a counterpart in harmonic oscillator properties. Thus, the total angular momentum contracts to the harmonic oscillator Hamiltonian, with the spherical harmonics and their properties contracting to the harmonic oscillator eigenfunctions (and corresponding properties).

APPENDIX I. DISENTANGLING THEOREM FOR ANGULAR MOMENTUM OPERATORS

In dealing with noncommuting exponential operators it is very useful to be able to change a symmetrized exponential operator into an ordered product of exponential operators. The well-known Baker-Campbell-Hausdorff formula (3.11) is of this type. Similar expressions can be obtained for angular momentum operators. We proceed to the derivation of these expressions by first considering the \(2 \times 2\) matrix representation of the rotation group,

\[
J_+ = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad J_- = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad J_z = \begin{pmatrix} 1/2 & 0 \\ 0 & -1/2 \end{pmatrix}
\]

which is the faithful group representation of smallest dimension.

By Maclaurin series expansion one finds

\[
\exp(w_+J_+ + w_-J_- + w_zJ_z) = \left( \begin{array}{cc} \cosh k + \frac{w_z}{2} \sinh k & \frac{\sinh k}{k} \\ \frac{w_+}{2} \sinh k & \cosh k - \frac{w_z}{2} \sinh k \end{array} \right)
\]  

(A1.1)

with

\[
k = (w_+w_- + w_z^2/4)^{1/2},
\]

and similarly

\[
\exp(x_+J_+) \exp[(\ln x_z)J_z] \exp(x_-J_-) = \left( \begin{array}{cc} \sqrt{x_z} + \frac{x_z}{\sqrt{x_z}} & \frac{x_z}{\sqrt{x_z}} \\ \frac{1}{\sqrt{x_z}} & \sqrt{x_z} \end{array} \right)
\]  

(A2.1)

\[
\exp(y_+J_+) \exp[(\ln y_z)J_z] \exp(y_-J_-) = \left( \begin{array}{cc} \sqrt{y_z} & y + \sqrt{y_z} \\ -\sqrt{y_z} & \frac{1}{\sqrt{y_z}} + y\sqrt{y_z} \end{array} \right)
\]  

(A3.1)

Equating these three matrices element by element gives expressions for each set of coefficients in terms of the others. This procedure gives four equations for three variables, but since the \(J\) matrices are traceless, the determinant of each group operation [(A1.1)-(A3.1)] is unity, therefore only three of these equations are independent. The applicability of the resulting operator equation

\[
\exp(w_+J_+ + w_-J_- + w_zJ_z) = \exp(x_+J_+) \exp[(\ln x_z)J_z] \exp(x_-J_-) = \exp(y_+J_+) \exp[(\ln y_z)J_z] \exp(y_-J_-)
\]  

(A4.1)

is not restricted to the \(2 \times 2\) matrix representation. The algebra of infinitesimal rotation operators maps onto the rotation group, which is represented by the exponential operators. Any relation between exponential
operators, i.e., between group operations, which is valid for one particular faithful representation of the group remains valid for all others. Therefore the equalities (A1.4) are general.

Using these relations, expressions for the rotation operator (4.7) are obtained:

\[
R_{\phi, \varphi} = \exp(\zeta J_+ - \zeta^* J_-) = \exp(\tau J_y) \exp[\ln(1 + |\tau|^2) J_z] \exp(-\tau^* J_+) \\
= \exp(-\tau^* J_y) \exp[-\ln(1 + |\tau|^2) J_z] \exp(\tau J_y)
\]

where \( \zeta \) and \( \tau \) are given in (4.7b) and (4.11b). If we let \( \zeta = c_\alpha, \zeta^* = c_\alpha^*, \alpha^* = c_\alpha^+, a = c_J, \) and \( J_z = a^+ a - 1/2 c_\alpha^2 \), following the contraction procedure of Sec. V, Eq. (A1.5) gives, in the limit \( c \to 0 \),

\[
T_\alpha = \exp(\alpha a^+ - a^* a) = \exp(\alpha a^+) \exp(-|\alpha|^2/2) \exp(-a^* a)
\]

\[
= \exp(-a^* a) \exp(|\alpha|^2/2) \exp(\alpha a^+)
\]

which is the Baker-Campbell-Hausdorff formula (3.11). A more general expression can be obtained by the contraction of (A1.4). If we let \( w_+ = c_\alpha, w_- = c_\beta^*, \) and \( w_z = 0 \), we obtain

\[
\exp(\alpha a^+ - \beta a^*) = \exp(\alpha a^+) \exp(-\beta a^*/2) \exp(-\beta^* a)
\]

\[
= \exp(-\beta^* a) \exp(\beta a^*/2) \exp(\alpha a^+)
\]

Using this relation, together with (A1.6), one obtains, after some manipulations

\[
T_\alpha T_\beta = \exp[(\alpha \beta^* - \alpha^* \beta)/2] T_{\alpha + \beta}
\]

which describes the composition of translations \( T_\alpha \). The use of this equation allows one to derive (3.14a) very simply. Of course, Eq. (A1.8) can also be obtained by contraction of a similar equation for the composition of rotations:

\[
R_{\phi', \varphi} R_{\phi, \varphi} = R_{\phi', \varphi} \exp(-i\psi J_z)
\]

where \( \phi, \psi, \) and \( \varphi \) are to be determined. We note that the \( R_{\phi, \varphi} \)'s do not form a group, since we have restricted ourselves to rotations around an axis in the \((x, y)\) plane. It is therefore necessary to allow for a rotation around the \( z \) axis on the right-hand side of (A1.9). This rotation simply amounts to changing the phase factor of the single atom eigenstates \( |\psi_i^R\rangle \).

The angles \( \phi, \psi, \) and \( \varphi \) could be obtained by manipulating (A1.4) and (A1.5). A simpler procedure is to use the \( 2 \times 2 \) matrix representation as in (A1.1)-(A1.3). By application of (A1.1) one has

\[
R_{\phi, \varphi} = \frac{1}{(1 + |\tau|^2)^{1/2}} \begin{pmatrix} 1 & \tau \\ -\tau^* & 1 \end{pmatrix}
\]

APPENDIX II. GENERATING FUNCTIONS FOR EXPECTATION VALUES WITHIN BLOCH STATES

Using the disentangling theorem of Appendix I together with the definition of Bloch states by the rotation (4.8), or equivalently (4.12), it is easy to construct generating functions for normally ordered, antinormally ordered, and symmetrized expectation values of products of powers of the operators \( J_x, J_y, J_z \) within Bloch states.

We define the following expectation values:

\[
X_N(\alpha, \beta, \gamma) = \langle \phi, \psi | e^{iJ_x} e^{iJ_y} e^{iJ_z} | \phi, \psi \rangle
\]

\[
X_A(\alpha, \beta, \gamma) = \langle \phi, \psi | e^{iJ_y} e^{iJ_z} e^{iJ_x} | \phi, \psi \rangle
\]

\[
X_S(\alpha, \beta, \gamma) = \langle \phi, \psi | e^{iJ_x + iJ_y + iJ_z} | \phi, \psi \rangle
\]

and will show that these functions can easily be calculated. These functions are generating functions since one has

\[
\left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] = \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] b^\dagger c^\dagger A_N^{-1} = \langle \phi, \psi | J_x^a J_y^b J_z^c | \phi, \psi \rangle
\]

\[
\left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] = \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] b^\dagger c^\dagger A_A^{-1} = \langle \phi, \psi | J_y^a J_z^b J_x^c | \phi, \psi \rangle
\]

\[
\left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] = \left[ \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right] b^\dagger c^\dagger X_S^{-1} = \langle \phi, \psi | S[J_x^a J_y^b J_z^c] | \phi, \psi \rangle
\]

where \( S[ ] \) means the fully symmetrized sum of products, which is equal to

\[
\text{APPENDIX II. GENERATING FUNCTIONS FOR EXPECTATION VALUES WITHIN BLOCH STATES
}

the sum of all distinct permutations of the factors within the bracket divided by the number $(a+b+c)!(abc)$ of these permutations.

It is sufficient to compute one of the generating functions, as the other two are then given by using the disentangling theorem of Appendix I. It is simplest to do this for $X_A$. Using (4.12) one has

$$X_A = \frac{1}{(1 + |\tau|^2)^{2J}} \langle -J | e^{(\tau^* + \gamma)J} e^{\delta J_z} e^{(\tau + \alpha)J} | -J \rangle \quad (\text{AII.3})$$

This expression is then put in normally ordered form (AII.2), in which case only the term $\exp[(\ln x_x)J_z]$ contributes to the expectation value in the ground state: $\langle -J | \exp[(\ln x_x)J_z] | -J \rangle = 1/x_x^J$. One obtains

$$\frac{1}{x_x} = \frac{1}{\sqrt{y_z}} + y_y \sqrt{y_z} = e^{-\delta/2} + (\tau + \alpha)(\tau^* + \gamma) e^{\delta/2} \quad (\text{AII.4})$$

which gives immediately

$$X_A = \left( e^{-\delta/2} + e^{\delta/2} (\tau + \alpha)(\tau^* + \gamma) \right)^{2J}$$

$$= (e^{-\delta/2} \cos^2(\delta/2) + e^{\delta/2} \sin(\delta/2) e^{-i\phi} + \alpha \cos(\delta/2))^{2J}$$

REFERENCES