Optical vortex beams are made to interact via degenerate two-wave mixing in a Kerr-like nonlinear medium. Vortex mixing is shown to occur inside the medium, leading to exchange of topological charge and cascaded generation of vortex beams. A mean-field model is developed and is shown to account for the selection rules of the topological charges observed after the wave-mixing process. Fractional charges are demonstrated to follow the same rules as for integer charges.

Optical vortices, or wave-front dislocations, have been identified as the singular points where the field goes to zero and around which the phase screws up as an n armed spiral, with n the topological charge [1–3]. When appearing in a large number, as after the propagation through a distorting medium, such phase singularities, also called topological defects, have been seen as disturbances imposing severe limitations to aberration correction systems [4]. Their statistical properties have been investigated also in nonlinear optical systems and shown to provide the scaling laws associated with the route toward space-time chaos [5]. More recently, the presence of single, or a few, optical vortices, appearing as phase singularities in low-order Gauss-Laguerre beams, has been revisited in view of useful applications, as the exchange of angular momentum between light and matter [6], realization of optical tweezers [7,8], quantum computation [9], and improvement of astronomical imaging [10].

Up to now, the controlled generation of optical vortex beams has been mainly realized by using linear methods, including the synthesis through holographic masks [11,12], the deformation of segmented adaptive mirrors [13], spiral glass plates [14] or pre-imposed director orientation in liquid-crystal samples, so-called, q-plates [15], or from micron-sized liquid-crystal droplets [16]. On the other hand, nonlinear vortex interaction has been investigated in a few experimental situations, namely, second-harmonic generation (SHG) [17] and parametric down-conversion in solid-state crystals [18], Raman-resonant four-wave mixing (4WM) in hydrogen gas [19], and nondegenerate 4WM in Rb atomic vapors [20]. Except for the parametric down-conversion, angular momentum conservation has been shown to be satisfied in the other cases, leading to selection rules of the type $l_{\text{SHG}} = 2l_a$ for SHG, where $l_{\text{SHG}}$ is the charge of the frequency doubled beam and $l_a$ that of the input beam; $l_A = 2l_p - l_S$ for resonant 4WM, where $l_A$ is the charge of the anti-Stokes beam, $l_p$ that of the pump, and $l_S$ that of the signal; and $l_S = l_F + l_B - l_p$, where $l_F$ and $l_B$ are the charges of, respectively, the forward and the backward beam, for the nondegenerate 4WM. From the theoretical point of view, nonlinear vortex interaction in wave-mixing processes has been predicted to induce a cascaded generation of vortices, leading to fundamental effects such as the generation of helical soliton beams in nonlinear optics [21] or Bose-Einstein condensation in two-dimensional wave turbulence [22].
the vortex beam with a homogeneous reference beam that has
an order being at an angle \( \sin \theta_m = m \lambda / L \) and carrying a topo-
logical charge \( m_{\text{hol}} \), with \( \lambda \) the wavelength of the input beam.
Moreover, in order to get a better diffraction efficiency on the
medium, on the examples shown in Fig. 3 it can be easily
verified that for \( l_a = 0, l_b = 1 \) we obtain \( l_{-2} = 2, l_{+1} = -1 \); while
for \( l_a = 1, l_b = -2 \) we get \( l_{-2} = -5, l_{+1} = 4 \).

Similar selection rules are observed also for input beams
with fractional charges. A fractional charge can be constructed
in the near field by imposing a half-fringe phase slippage
over one half of the interferometric hologram. However, it
is unstable upon propagation in free space, because it
transforms into a charge with a value equal to its nearest
integer plus an infinite array of unitary charges of alternate
sign [26,27]. Correspondingly, the field amplitude goes to
zero, generating a characteristic black line in the intensity
profile. Examples of the interaction between vortex beams with
fractional charges are shown in Fig. 4 for \( l_a = 0, l_b = +1/2 \)
in Fig. 4(a) and \( l_a = +1, l_b = +1/2 \) in Fig. 4(b). The output
charges are \( l_{-2} = 1, l_{+1} = -1/2 \) and \( l_{-2} = 0, l_{+1} = +3/2 \),

![Image](https://example.com/image1.png)
losses.

a weak vortex beam, as previously synthesized through a zero topological charge pump could be used to amplify intensity much higher than the signal intensity. In particular, performed in the classical pump-signal scheme, with the pump

However, it is worth noting that vortex mixing can in general be better revealing the output selection rules, vortex mixing [25]. In the present set of experiments, for the purpose of intensity of the other beam, the signal to be amplified

cancellation results clearly from the interference image, the cancellation of the topological charge. Even though this zeros disappear upon propagation and are totally absent in the far field.

Note that two-wave mixing in the LCLV can lead to optical amplification, when the intensity of one of the two beams, usually called the pump, is much higher than the intensity of the other beam, the signal to be amplified [25]. In the present set of experiments, for the purpose of better revealing the output selection rules, vortex mixing was performed with two beams of almost equal intensity. However, it is worth noting that vortex mixing can in general be performed in the classical pump-signal scheme, with the pump intensity much higher than the signal intensity. In particular, a zero topological charge pump could be used to amplify a weak vortex beam, as previously synthesized through a holographic mask or other methods often introducing intensity losses.

The theoretical description can be developed by considering the nonlinear Schrödinger equation [28]

\[ \frac{\partial A}{\partial z} = - \left( \frac{1}{2k_0n_0} \nabla_\perp^2 + n_2 k |A|^2 \right) A, \tag{1} \]

where \( k_0 = 2\pi/\lambda_0 \) is the optical wave vector, \( A \) is the slowly varying amplitude of the optical field \( E = AE^{(k_0z-\omega t)} \), \( n_0 = 1.6 \) the constant part of the refractive index, and \( n_2 = -7 \text{ cm}^2/\text{W} \) the Kerr-like coefficient of the LCLV [25]. The amplitude of the optical field before the interaction is the sum of the two vortex beams \( A_a \) and \( A_b \), carrying, respectively, the

topological charges \( l_a \) and \( l_b \), that is, \( |A(\vec{r}, z = 0)|^2 = |A_a|^2 + |A_b|^2 + 2A_aA_b \cos(\vec{K}_g \cdot \vec{r} + (\ell_b - \ell_a)\phi) \), the two beams induce a phase grating with spatial period \( \Lambda = 2\pi/\vec{K}_g \) and with a dislocation given by the difference \( \ell_b - \ell_a \). In the usual experimental conditions, \( \Lambda \gg d \), so the beam coupling can be treated in the Raman-Nath regime of diffraction; hence, the field at the exit of the NL medium can be analytically calculated and is given by [25]

\[ E(\vec{r}, z = d) = \sum_{m=-\infty}^{+\infty} A_m e^{i[l_k \vec{r} + \omega_t - \omega t] + \text{c.c.}} \tag{2} \]

that is, it is made up of a series of diffracted orders, the amplitude of each is given by \( A_m = r^m [A_a J_m(\rho) + rA_b J_m(\rho)] e^{jkd\sin(\theta_0 + \pi/2) + j\pi/2} \), \( J_m \), the Bessel function of the first kind and of order \( m \) and \( \rho = 2k_0dA_aA_b \), and each is characterized by the wave vector

\[ \vec{k}_m = \vec{k}_a - m\vec{K}_g, \tag{3} \]

which expresses the momentum conservation, and by the topological charge

\[ \ell_m = \ell_a - m(\ell_b - \ell_a) \tag{4} \]

which expresses the angular momentum conservation.

Equation (4) gives the selection rules on each output order, nicely accounting for the experimental observations. The 0 and \( -1 \) orders, propagating parallel to \( E_a \) and, respectively, \( E_b \), maintain the same topological charge. On the other hand, the topological charge of the +1 and \( -2 \) orders is, respectively, \((2\ell_a - \ell_b)\) and \((2\ell_b - \ell_a)\), that is, during the vortex mixing the topological charge is exchanged on the diffracted beams. Experimentally, we have verified that the charge observed on
the outer orders up to \( m = \pm 3 \) coincides with the theoretical predictions. Note, also, that the vortex interaction generates a cascade of topological charges, with multiple vortex beams at the output. For instance, if we consider the +1 and −2 orders, we have the same \( l_a + l_b \) topological charge as at the input, but taking also the −2 and +2 orders, we get \( 2(l_a + l_b) \), that is, the double of the input charge.

Finally, we have verified that the diffusive term characterizing the NL medium response, and neglected in the mean-field approximation, does not influence the topological charge of the output orders. Indeed Eq. (1) should be coupled with a spatial relaxation equation for the refractive index \( \nabla^2 n = n_0 + n_2 |E|^2 \), where \( l_d \sim 5 \; \mu \text{m} \) is the transverse diffusion length in the NL medium [25]. When inserted into Eq. (1), the above expression allows us to arrive at the same result as before, except that the shape of the vortex is slightly smoothed by a weak correction provided by the diffusion term. However, the output charge remains unaltered for all the output orders.

In conclusion, we have shown that vortex mixing can be performed in a Kerr-like nonlinear medium. The selection rules for the output charges are theoretically identified and experimentally demonstrated. The obtained cascade of topological charges is particularly interesting for vortex control applications, such as multiplication of the topological charge and generation of new vortex beams, whereas vortex beam amplification could be obtained by exploiting the gain feature of the wave-mixing process.

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