GENERALIZED FOKKER-PLANCK EQUATION FOR A NONLINEAR BROWNIAN MOTION WITH FLUCTUATIONS IN THE CONTROL PARAMETER ☆

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We derive a generalized Fokker-Planck equation which account not only for the fluctuations of the internal parameter describing a statistical system, but also for fluctuations in the control parameter which may induce rapid changes in the force law. As an example, the Fokker-Planck equation for the optical bistability is derived in its full extent by simple heuristic considerations.

1. Introduction

Quantum optical transition phenomena as the laser threshold or the optical bistability can be described by a mean field approach, in terms of a single order parameter x (the e.m. field) driven by a nonlinear force depending by some control parameters y (e.g., the excitation rate of active atoms in the laser, the density of absorbing atoms or the amplitude of the external field in the optical bistability [1–3].

We report here a generalized Fokker-Planck equation for the probability density $P(x,t)$ of the internal parameter x, when the external parameter y is affected by fluctuations. The treatment is of general validity for all transition phenomena (either pumped or at thermal equilibrium), whenever they are described by a mean field. As an example we rederive in a simple way by heuristic arguments the nonlinear diffusive equation for the optical bistability previously described by quantum mechanical arguments [4].

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2. The modified Fokker-Planck equation

We recall that the standard theory of the Brownian motion [5] generalized [1] to nonlinear systems, is given in terms of the Langevin equation

$$\dot{x} = f(x,y) + F \tag{1}$$

where x and y have been previously defined, $f(x,y)$ is the nonlinear deterministic force, and F is a gaussian stochastic process, with zero average and δ-correlated in time, accounting for the coupling of x to a thermal reservoir responsible for the Brownian motion.

With the above assumptions, eq. (1) gives rise to a Fokker-Planck equation for the probability density $P(x,t)$

$$\partial P/\partial t + \partial AP/\partial x + \tfrac{1}{2}\,\partial^2 BP/\partial x^2 = 0, \tag{2}$$

where

$$A = \langle \Delta x \rangle/\Delta t, \qquad B = \langle \Delta x^2 \rangle/\Delta t \tag{3, 4}$$

are the ensemble average of the increments of x and x^2 in the limit of a vanishing time interval Δt. It can be shown [5,1] that if $2D$ is the correlation amplitude of F, then $A = f(x,y)$ and $B = 2D$ and hence the above equation becomes

$$\partial P/\partial t + \partial f(x,y)\,P/\partial x + \partial^2\,\Delta P/\partial x^2 = 0. \tag{5}$$

Let us consider now a fluctuating control parameter. This may be the case of a para-ferromagnetic transition in the presence of a magnetic field H affected by noise, or the optical bistability either with a noisy driving field or with a fluctuation in the population difference. We replace y by $y + \delta y$ and take δy as a zero mean random process.

For simplicity we consider the case of a rapidly fluctuating δy ‡ with respect to the time scale t_d of the deterministic evolution

$$\langle \delta y(0)\, \delta y(t)\rangle = 0 \qquad \text{if } t > t_c$$

$$\langle \delta y(0)\, \delta y(t)\rangle = 2Q/t_c \qquad \text{if } t < t_c, \tag{6}$$

with $t_c \ll t_d$, where t_c is the correlation time of δy. Further we assume δy uncorrelated with F.

The Langevin equation then becomes

$$\dot{x} = f(x,y) + \frac{\partial f}{\partial y}\delta y + \tfrac{1}{2}\frac{\partial^2 f}{\partial y^2}\delta y^2 + \ldots + F. \tag{7}$$

By integrating this equation over a small interval Δt and taking the average with the above assumptions, it is easy to show that first average A is, neglecting higher than second order terms

$$A = f(x,y) + \frac{\partial^2 f}{\partial x \partial y}\frac{Q}{t_c}, \tag{8}$$

while the second average B is modified as

$$B = 2D + 2Q(\partial f/\partial y)^2. \tag{9}$$

Hence, the new generalized Fokker-Planck equation is

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x}\left(f + \frac{\partial^2 f}{\partial x \partial y}\frac{Q}{t_c}\right)P + \frac{\partial^2}{\partial x^2}\left(D + Q\left(\frac{\partial f}{\partial y}\right)^2\right)P. \tag{10}$$

Eq. (10) is a general feature of statistical systems described by $(n + 1)$ parameters x, $\{y\}$ ($\{y\} = y_1, y_2, \ldots y_n$) whenever the $\{y\}$ are correlated on a time scale faster than that of x, as assumed for a single y in eq.(6).

The corrections included in eq. (10) with respect to eq. (5) may be so small that they cannot be observed with respect to the leading terms. This is the case of the laser. There are situations however where the two terms are comparable. This is the case of the optical bistability as shown in the next section.

‡ Such was for instance the loss-switch in the transient laser fluctuation experiments [6].

3. Application to quantum optical transition phenomena

We apply the above eq. (10) to the case of the absorptive optical bistability.

We start from the equation given in refs. [2–4]

$$\dot{x} = k(y - x - 2Cx/(1 + x^2)). \tag{11}$$

We recall that this equation describes the evolution of the field x transmitted by a lossy medium at resonance with an incident field y and put in a cavity whose loss rate is k.

The fields are suitably normalized so that the "cooperation number" C is given by

$$2C = \gamma_c^2/k\gamma_\perp, \tag{12}$$

where

$$\gamma_c^2 = (\omega\mu^2/\hbar\epsilon)N = g^2 N \tag{13}$$

is the cooperative rate introduced in ref. [7] which is proportional to the atomic density N, and $\gamma_\perp$ is the decay rate of the induced atomic polarization.

We now evaluate the diffusive term to be put into a fenomenological Fokker-Planck equation for fluctuations in the cooperation parameter C. From eqs. (12), (13) we have

$$\delta C = (g^2/k\gamma_\perp)\,\delta N, \tag{14}$$

where δN is the fluctuation in the population difference between lower and upper state. If σ_i is the ith atom population difference and we consider uncorrelated variations of these parameters ($\langle \delta\sigma_i \delta\sigma_j\rangle = 0$ for any time, if $i \neq j$) the correlation of δN will be

$$\langle \delta N(0)\, \delta N(\tau)\rangle = N\langle \sigma_i(0)\, \sigma_i(\tau)\rangle$$

$$= \frac{N}{\gamma_\parallel}\Delta(\tau), \qquad \int_{-\infty}^{+\infty} \Delta(\tau)\, d\tau = 1, \tag{15}$$

where $\Delta(\tau)$ is a normalized time function lasting for a correlation time so short that for the present purposes it can be assimilated to a δ-function, and $\gamma_\parallel$ is the longitudinal atomic decay rate.

Going now to eq. (10) and specialising it with respect to the force equation (12), where the parameter to be varied is C, we realize that:

i) There is no extra contribution to the drift term, since $\partial^2 f/\partial c^2 = 0$.

ii) The diffusion term D must be equal to zero. Indeed in the laser case, it would be proportional to the spontaneous noise on the lasing mode due to the equilibrium population of the upper atomic state [8]. But here, in the absence of the incident field, we have no spontaneous noise, since the atoms are in the ground state.

iii) The correlation amplitude of δC is given in terms of eqs. (14), (15) by

$$Q = g^4 N / 4k^2 \gamma_\perp^2 \gamma_\parallel . \tag{16}$$

iv) The above correlation amplitude has to be weighted by

$$(\partial f/\partial c)^2 = 4k^2 x^2/(1+x^2)^2, \tag{17}$$

where use was made of eq. (11).

Combining eqs. (16) and (17) into eq. (10) we obtain the following optical bistability Fokker-Planck equation

$$\frac{\partial P}{\partial t} + \frac{\partial}{\partial x} k \left(y - x - \frac{2Cx}{1+x^2} \right) P + \frac{g^4 N}{\gamma_\perp^2 \gamma_\parallel} \frac{\partial^2}{\partial x^2} \left(\frac{x^2}{(1+x^2)^2} P \right) = 0, \tag{18}$$

which coincides with eq. (4.2) of ref. [4], there derived by a general quantum statistical formalism.

4. Conclusion

We have derived the optical-bistability statistical equation by the joint use of the general Fokker-Planck equation and some heuristic arguments.

Eq. (10) is relevant for all Landau type mean field transitions, whenever fluctuations do not stem from the spreading of the order parameter within a definite potential well, but they rather stem from the wandering through a manifold of different potential curves, that is, whenever the fluctuations affect one of the control parameters.

We may ask why this important point was not drawn previously. In our opinion the reason is due to the fact that, whenever one modifies a Landau free energy by a term bilinear in the order parameter x and in a control parameter y, the only effect in eq. (5) is an additional constant diffusive contribution (just $D + Q$ in our eq. (10)).

Such is the case, for instance of the mean field treatment of a magnetic phase transition in the presence of an H field, where the extra term in the free energy is (M being the magnetization)

$$\Delta F = - M \cdot H .$$

That would also be the case, had we introduced fluctuations in the incident field y of eq. (11). Such more elementary case was dealt with recently by Schenzle [9].

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