A NEW DESCRIPTION OF THE DECAY OF AN UNSTABLE STATE

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We introduce a new parameter to describe the decay of an unstable state, that is, the time spread of the trajectories at the intersection of a given threshold. The constancy of the "time spread" is a necessary condition for a simplified two-piece (linear diffusive plus nonlinear deterministic) description of a decay.

An experiment on the time dependent photon statistics of a laser [1] opened a long lasting discussion on transient statistics. This is a particularly relevant subject in so far as the decay of an unstable state is accompanied by anomalous fluctuations which do not scale with the inverse of the system size, and hence make thermodynamic treatment in terms of averaged values insufficient to deal with the phenomenon.

A satisfactory approach introduced for the laser [2] and then extended to more general cases [3,4] consists in first letting the system decay from the unstable point under the linearized part of the deterministic force, diffusing simultaneously because of the stochastic forces. This leads to a short time probability distribution of easy evaluation. Then we solve for the nonlinear deterministic path and spread it over the ensemble of intial conditions previously evaluated in the linear regime.

A non-piecewise treatment has been suggested recently [5,6] but it is strongly model dependent as shown in a recent generalization of those approaches [7].

If x is the parameter describing the system it has been customary to represent the anomalous fluctuations in terms of the variance $\langle\Delta x^2(t)\rangle$, where the average is taken over an ensemble of classical trajectories, solutions of the deterministic equation

$$\dot{x} = F(x) , \tag{1}$$

whose density is weighted by the probability density $P(x_0)$ of the initial condition x_0. Notice that x_0 does not necessarily have to be $x(t = 0)$ but it is better evaluated at the end, $t = t_0$, of the linearized region.

Here we want to show that the quantity $\langle\Delta x^2\rangle$ is not statistically relevant and we replace it with a more meaningful quantity.

Indeed an ensemble average has a relevance in a stationary ergodic ensemble where it can replace a time average on a specific physical trajectory. In the decay of an unstable state, measuring $\langle\Delta x^2\rangle$ means to repeat a measurement of x at a fixed time distance from the onset of the decay and evaluate the spread over different trajectories [2]. Hence $\langle\Delta x^2\rangle$ does not contain any further statistical information beyond that already included in the initial distribution $P(x_0)$, provided the trajectories are deterministic.

The fact that $\langle\Delta x^2\rangle$ versus time goes through a high peak simply reflects the x dependence of the force $F(x)$. This is clarified in fig. 1 which is a qualitative plot taken from the experiments of refs. [1,2]. In (a) we have represented two deterministic trajectories which occur with a weight given by the distribution $P(x_0)$ plotted vertically on the left side of the diagram. Since for $t > t_0$ we do not consider statistical effects, the two trajectories are just shifted in time by a constant amount Δt. Hence, at each time t the difference in height Δx is given by

$$\Delta x = \dot{x}\,\Delta t = F(x)\Delta t \tag{2}$$

(the dot denotes a time derivative) and therefore the excess of variance with respect to the initial variance $\langle \Delta x_0^2 \rangle$ reflects for each time the local force value.

These considerations show that a more relevant parameter is the spread in the times at which the different trajectories cross a fixed x value. This corresponds to a well defined laboratory operation. Furthermore, as explained above, this is a parameter whose fluctuations are related to the presence of non trivial fluctuations at all times, that is, to a statistical spread within each trajectory and not only in the initial manifold.

In view of the above considerations, we define a nonlinear transformation

$$d\tau = dx/F(x)\,. \tag{3}$$

We start from a general Fokker–Planck equation (FPE) for the stochastic process $x(t)$, including possibly an x dependent diffusion coefficient D

$$\partial P/\partial t = -\,\partial FP/\partial x + \partial^2 DP/\partial x^2\,. \tag{4}$$

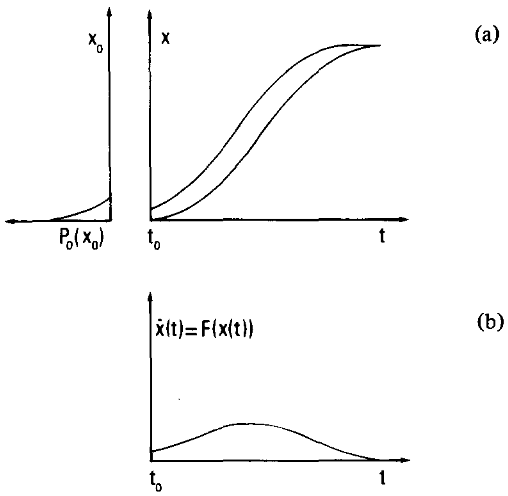

Fig. 1. (a) A qualitative plot of two deterministic trajectories and of the weight function of the initial condition. (b) Corresponding plot of the deterministic force.

The associated FPE for the transformed quantity τ is

$$\partial Q/\partial t = -\,(\partial/\partial\tau)(1 - DF'/F^2)Q + (\partial^2/\partial\tau^2)DQ/F^2\,, \tag{5}$$

where $F' \equiv dF/dx$ is evaluated at $x = x(\tau)$.

The second cumulant $K_2 = \langle\tau^2\rangle - \langle\tau\rangle^2$ of the Q distribution obeys the rate equation (written for simplicity for a constant D)

$$\dot{K}_2 = D(2\langle F'/F^2\rangle\langle\tau\rangle - \langle F'\tau/F^2\rangle) + 2D\langle 1/F^2\rangle\,. \tag{6}$$

Notice that the rate $\dot{K}^2$ is proportional to D. Now the previous treatments are based on a deterministic evolution in the nonlinear regime, which amounts to cancelling the last term of eq. (4) and hence to have K_2 = const. as shown qualitatively in fig. 1.

As a further comment, eq. (6) offers the necessary condition for the qualitative piecewise procedure of joining a short time linearized diffusion to a long time deterministic motion. Indeed when F gets very large (see fig. 1b) it is legitimate to take $\dot{K}_2 = 0$ in eq. (6) and hence to neglect the diffusion term in the FPE (4).

Consider now a nonconstant diffusion

$$D = D_0 + \eta x^2\,. \tag{7}$$

Let us restrict ourselves to the first region of the instability where the force can be linearized as $F = \gamma x$. Once F has become large enough to make the D_0 contribution negligible the FPE for our time spread reduces to

$$\partial Q/\partial t = -\,(\partial/\partial\tau)(1 - \eta/\gamma)Q + (\eta/\gamma^2)\,/\partial^2 Q/\partial\tau^2\,. \tag{8}$$

Hence the spread in time becomes

$$\langle\Delta\tau^2\rangle = K_2^0 + (2\eta/\gamma)t\,, \tag{9}$$

where K_2^0 is the constant cumulant due to D_0, which can be evaluated with the usual piecewise technique. Eq. (9) shows that the trajectories are no longer rigidly shifted with respect to one another, but that they undergo a time dependent distortion.

A nonconstant diffusion occurs in many physical problems as optical bistability [8,9] or a class of electronic instabilities discussed recently both experimentally [10] and theoretically [11]. Therefore one should expect a time dependent spread among trajectories, besides the constant shift consequent to the initial conditions.

In conclusion, we must emphasize that the operational definition taken as a starting point (τ as the spread in the crossing of a given x) is only approximately represented by the mathematical definition (3). Indeed eq. (3) is a nonlinear mapping which yields a spread depurated of the trivial dependence on the deterministic force, and hence easily connected to the diffusion D [see eq. (6)]. A definition according to the crossing idea should rely on a modified eq. (3) including a stochastic Langevin force, and lead to a FPE in which the ordering parameter is x and *not* t. This difficult task will be presented in a forthcoming paper.

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