A HEURISTIC DERIVATION OF THE TIME DEPENDENT PROPERTIES OF A FREE ELECTRON LASER

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We analyze the behaviour of the laser pulse in a Free Electron Laser during a single pass in the small signal limit. A space-time picture yields in a simple way the correct boundary conditions to solve the equations. The meaning of lethargy is also heuristically analyzed.

1. Introduction

After the initial proposals [1] and the first experimental realization of a Free Electron Laser (FEL) [2] interest has arisen on the FEL transient behaviour in the multimode regime. Some authors have treated the problem in the framework of a multiparticle analysis [3] by studying the coupled Maxwell and Boltzmann equations for the laser slowly varying envelope and the electron distribution function [4]. Other authors have treated the problem on the basis of a single particle multimode analysis of the FEL in a storage ring [5] and single pass regime [6,7], studying a closed equation for the stationary electric field envelope [6].

The most interesting feature of the obtained results is the existence of a behaviour similar to laser lethargy, namely the FEL gain depends on the mismatch $\delta T = T_c - T_e$ between the electron bunches repetition time T_e and the cavity round-trip T_c and it is maximum for a particular $\delta T \neq 0$.

The aim of the present paper is to discuss a heuristic approach to the FEL transient multimode behaviour and the physical meaning of FEL lethargy.

By reference to a world line crossings we can easily localize the interaction regions among electrons, laser field and wiggler field, thus assigning the suitable boundary conditions to solve the time dependent equations.

2. General formalism

The description is made in the reference frame where the wiggler and laser fields have the same frequency and the electron beam (assumed monochromatic in energy) has an initial velocity $v_0 \ll c$ [8]. We assume the following expression for the vector potential A

$$A = [A_L(z,t)\exp\{-i(\omega t - kz)\} + A_W(z,t)\exp\{-i(\omega t + kz)\}]\,e_- + \text{c.c.}\,, \tag{1}$$

$$e_\pm = e_x \pm i e_y\,, \quad \omega = kc\,,$$

where A_W corresponds to a square pulse, that is,

$$A_W = \begin{cases} \bar{A}_W = \text{const.} & 0 \leqslant z + ct \leqslant L_W\,, \\ 0 & \text{otherwise}\,, \end{cases} \tag{2}$$

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and $A_L \ll A_W$ is a slowly varying function. The field evolution is given by the Maxwell equation

$$\Box A = -(4\pi/c)J, \quad J = en\mathbf{v}_t, \tag{3}$$

where v_t is the mean transverse velocity function. From (1), (2) and (3) we obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)A_L = \frac{2\pi r_0 c}{ik} A_W n\, e^{-2ikz},$$

$$r_0 = e^2/mc^2, \tag{4}$$

where n is the electron space density function. Both n and the electron mean longitudinal velocity function v, are defined in terms of the electron distribution function $h(z, u, t)$, as

$$n(z, t) = \int h(z, u, t)\,du,$$
$$nv(z, t) = \int u h(z, u, t)\,du. \tag{5}$$

The equations of motion for the electron beam can be reduced, in the "cold plasma" limit, to the equations for n and v, that is,

$$\partial n/\partial t + \partial(nv)/\partial z = 0,$$
$$\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial z} = -\frac{r_0}{m}\frac{\partial(\frac{1}{2}A^2)}{\partial z}. \tag{6}$$

Eqs. (6) are a system of nonlinear first order partial differential equations. We can reduce it to a linear one in the small signal limit with the following position

$$n = n_0(z, t) + [n_1(z, t)\,e^{2ikz} + \text{c.c.}],$$
$$v = v_0 + [v_1(z, t)\,e^{2ikz} + \text{c.c.}], \tag{7}$$

where $n_0, n_1 \ll n_0$ and $v_1 \ll v_0$ are slowly varying functions. With this substitution we finally obtain

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial z}\right)A_L = \frac{2\pi r_0 c}{ik} A_W n_1(z, t),$$

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)n_1 = -2ik(v_0 n_1 + v_1 n_0), \tag{8}$$

$$\left(\frac{\partial}{\partial t} + v_0\frac{\partial}{\partial z}\right)v_1 = -2ikv_0 v_1 - 4ik\frac{r_0}{m}A_L A_W.$$

3. Heuristic approach

Eqs. (8) make a system of linear, partial differential equations which can be, in principle, integrated exactly. We prefer first to discuss heuristically the laser pulse evolution during a single pass.

Assume the following simplified conditions

$$n_0(z, t) = \begin{cases} \text{const.} & 0 \leqslant z - v_0 t \leqslant -\sigma_z, \\ 0 & \text{otherwise}, \end{cases} \tag{9}$$

and refer to fig. 1, where we have depicted the evolution of the system in the space-time. The wiggler field is "seen" by the electrons as a square pulse running with velocity $-c$ (in the Weiszäcker–Willimas approximation), the laser envelope is running with velocity c and the electron bunch has a velocity $v_0 \ll c$. The interaction region is the intersection between the fields and the electron regions.

Consider a "ray" of the laser pulse which, at $t = 0$, is at z_0. This point enters in the wiggler at $P_1 \equiv (z_1, t_1)$, goes out the wiggler at $P_2 \equiv (z_2, t_2)$, enters in the electron beam at $P_3 \equiv (z_3, t_3)$ and goes out the electron beam at $P_4 \equiv (z_4, t_4)$ where

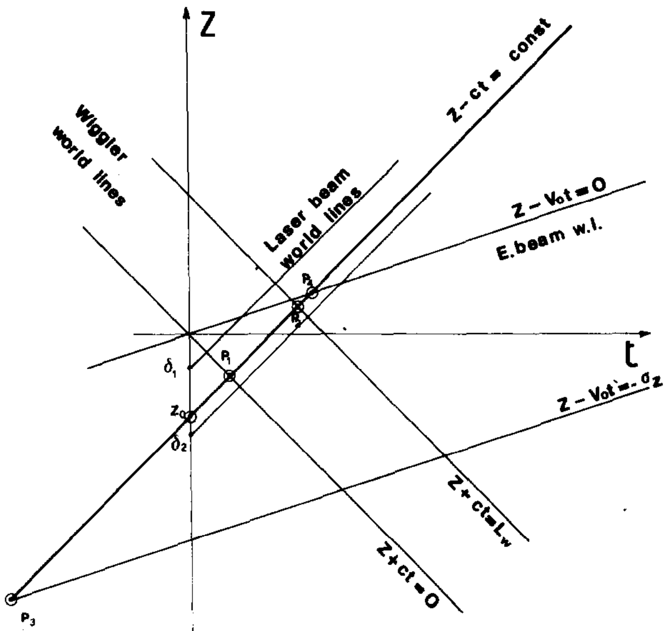

Fig. 1. World lines of laser, wiggler fields and electron density; we have considered a square laser pulse limited between $z_0 = \delta_1$ and $z_0 = \delta_2$.

$$z_1 = \tfrac{1}{2} z_0 , \qquad t_1 = -\frac{1}{2c} z_0 ,$$

$$z_2 = \tfrac{1}{2}(L_W + z_0), \qquad t_2 = \frac{1}{2c}(L_W - z_0)$$

$$z_3 = -\sigma_z - \frac{v_0}{c - v_0}(\sigma_z + z_0), \quad t_3 = -\frac{1}{c - v_0}(\sigma_z + z_0), \tag{10}$$

$$z_4 = -\frac{v_0}{c - v_0} z_0 , \qquad t_4 = -\frac{1}{c - v_0} z_0 .$$

The interaction time Δt of this ray is given by

$$\Delta t = \min[t_2, t_4] - \max[t_1, t_3] , \tag{11}$$

where

$$t_2 > t_4 , \quad z_0 > -\frac{c - v_0}{c + v_0} L_W ,$$

$$t_1 > t_3 , \quad z_0 > -\frac{2c}{c + v_0}\sigma_z . \tag{12}$$

Defining m_1 and m_2 as $z_0 = m_1$ when $t_2 = t_4$, and $z_0 = m_2$ when $t_1 = t_3$, we have that the difference $m_1 - m_2$ is given by

$$m_1 - m_2 = \frac{2c}{c + v_0}\sigma_z - (c - v_0)\frac{L_W}{c + v_0} = \frac{2c}{c + v_0}\sigma_z - \Delta . \tag{13}$$

Δ is the vacuum-phase slippage [5], i.e. is the maximum length between the laser pulse and the electron bunch, at the end of the interaction region, due to the different velocities.

To make further comments about the FEL transient behaviour we will suppose valid the single mode small signal gain formula. This is, in general, not consistent if $[2c/(c + v_0)]\sigma_z \approx \Delta$ (as in the Stanford experiment [2]): in this case we would consider the multimode small signal gain formula. So we suppose that $[(2c/(c + v_0)]\sigma_z \gg \Delta$, a condition which is well fulfilled, for example, in the storage ring operation. Of course, in the storage ring operation we must treat in a different way the electron-beam evolution (see for example ref. [5]).

From fig. 1 one can easily see that the rays of the laser envelope such that $z_0 > m_1$ interact for a time $\Delta t^{(1)}(z_0)$ given by

$$\Delta t^{(1)}(z_0) = -\frac{c + v_0}{2c(c - v_0)} z_0 , \tag{14}$$

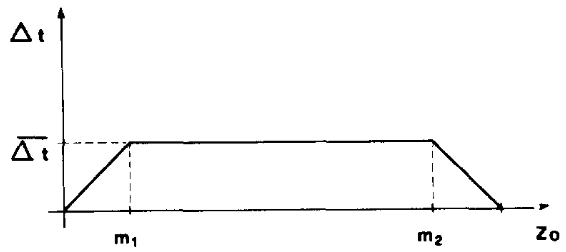

Fig. 2. Interaction time Δt of a point of the laser pulse with initial position z_0 versus z_0; $\overline{\Delta t}$ is the maximum interaction time.

the rays such that $m_2 \leqslant z_0 \leqslant m_1$ for a time $\Delta t = L_W/2c = [(c + v_0)/2c]\overline{\Delta t}$ and the rays such that $z_0 \geqslant m_2$ for a time $\Delta t^{(2)}$ given by (fig. 2)

$$\Delta t^{(2)}(z_0) = \overline{\Delta t}\frac{c + v_0}{2c} + \frac{\sigma_z}{c - v_0} + \frac{c + v_0}{2c(c - v_0)} z_0 . \tag{15}$$

The small signal gain versus Δt is given by [8] (fig. 3)

$$G = -B\frac{1}{\eta_0^3} x^3 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sin x}{x}\right)^2 , \tag{16}$$

where

$$x = \eta_0 \Delta t/\overline{\Delta t} , \quad \eta_0 = 2\omega\,\Delta t\, v_0/c ,$$

$$B = 32\pi^2 \frac{(r_0 c)^2}{m}\frac{N I_W(0)}{\Sigma_L \sigma_z V}(\Delta t)^3 .$$

Here, V is the mode volume, $I_W(0)$ is the initial wiggler field action, N is the total number of electrons, Σ_L is the laser beam cross section and σ_z is the electron bunch length. G as a function of z_0 is given in fig. 4a where we have supposed $\eta_0 \approx 1.3$. Now suppose to have a laser pulse as in fig. 4b. At the end of the interaction region we have

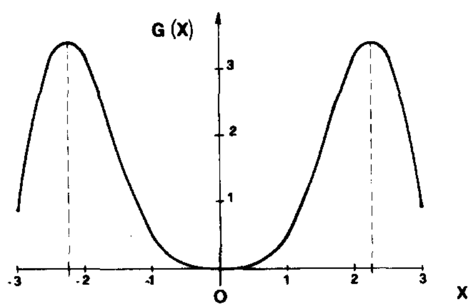

Fig. 3. Gain function versus the normalized interaction time $x = \eta_0\,\Delta t/\overline{\Delta t}$.

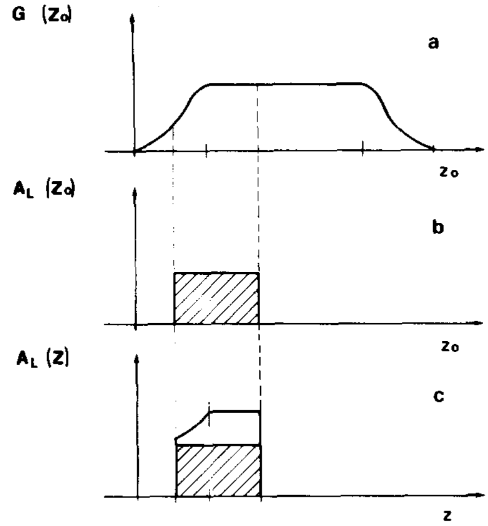

Fig. 4. Amplification of a laser square pulse a) gain function versus initial position of the point of the laser pulse, b) initial laser pulse, c) laser pulse at the end of the interaction region.

$$A_L(z, t > t_4) = A_L(z - ct) + G(z_0 - ct)A_L(z - ct),$$
$$A_L(x) = A_L(z = x, t = 0) . \tag{17}$$

From fig. 4c one can see that the "center" of the initial beam laser envelope (in a reference frame which moves with the laser pulse) is backshifted by the interaction. If one wants to maximize the gain in the next pass, one must adjust the electron bunch repetition time T_e and the cavity round trip time T_c so that the next electron bunch is also backshifted and the maximum laser field points can "explore" as much as possible of the electron bunch. This behaviour is the FEL lethargy found by the authors of refs. [4–6]. Similar behaviour is shown in fig. 5 for a gaussian pulse.

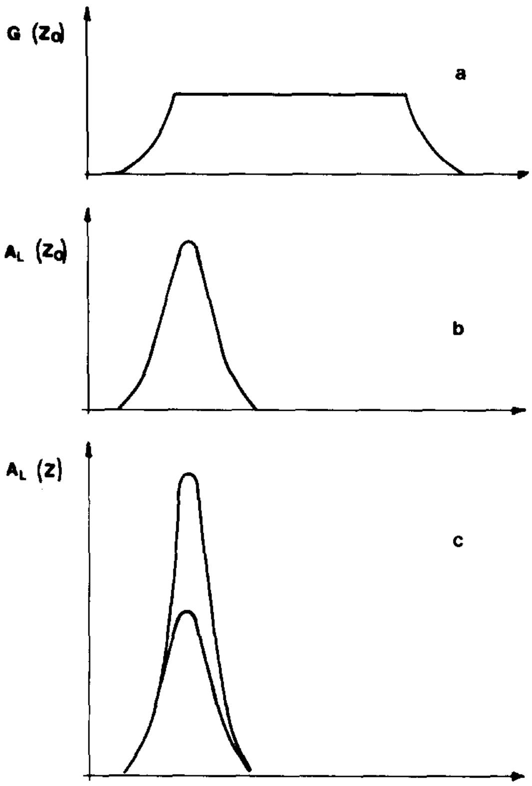

Fig. 5. Same as fig. 4 for a gaussian laser pulse.

The picture here offered uses a gain formula valid for a monochromatic field. However the behaviour is not qualitatively different even when the multimode structure is taken into account.

4. Analytical results

System (8) can be easily integrated firstly by introducing the new variables η and ξ defined by

$$\eta = z - v_0 t, \quad \xi = z/v_0 , \tag{18}$$

in term of which, and assuming the initial conditions $n_1(z, 0) = 0, v_1(z, 0) = 0$ the solutions of (8b) and (8c) are

$$v_1(\eta, \xi) = -4\,\mathrm{i}k\frac{r_0}{m}\exp(-2\mathrm{i}kv_0\xi)$$
$$\times \int_{-\infty}^{\xi} \mathrm{d}\xi' \exp(2\mathrm{i}kv_0\xi')A_L(\eta, \xi')A_W(\eta, \xi'),$$
$$n_1(\eta, \xi) = -8k^2\frac{r_0}{m}\exp(-2\mathrm{i}kv_0\xi)\int_{-\infty}^{\xi}\mathrm{d}\xi'\, n(\eta)$$
$$\times \int_{-\infty}^{\xi'} \mathrm{d}\xi'' \exp(2\mathrm{i}kv_0\xi'')A_L(\eta, \xi'')A_W(\eta, \xi'') . \tag{19}$$

Introducing the new variable ζ and τ defined by

$$\zeta = z - ct, \quad \tau = z/c , \tag{20}$$

we finally obtain from eq. (8a)

$$A_L(\zeta, \tau) = A_L(\zeta) + \mathrm{i}\frac{BV}{2\pi}F(\zeta, \tau) , \tag{21}$$

where

$$F(\zeta,\tau)=\int_{-\infty}^{\tau}\mathrm{d}\tau'\vartheta_W(\zeta,\tau')\exp(-2\mathrm{i}\omega\tau')\int_{-\infty}^{\xi''}\mathrm{d}\xi''\frac{n_0(\eta)}{N}$$
$$\times\int_{-\infty}^{\xi''}\mathrm{d}\xi'''\exp(-2\mathrm{i}kv_0\xi''')A_L(\xi''',\eta)\vartheta_W(\xi''',\eta),\qquad(22)$$

and

$$\vartheta_W(\zeta,\tau)=\begin{cases}1 & 0\leqslant 2c\tau-\zeta\leqslant L_W\,,\\ 0 & \text{otherwise}\,.\end{cases}\qquad(23)$$

We can assume A_L in $F(\zeta,\tau)$ to be a function only of $z-ct$, i.e.

$$A_L(\xi,\eta)=A_L[(c/v_0)\eta-(c-v_0)\xi]=A_L(\zeta)\,,\qquad(24)$$

and by Fourier transforming A_L and n_0, with some passages we finally obtain for $\tau\geqslant(L_W+\zeta)/2c$

$$F(\zeta)=\frac{c+v_0}{2c}\frac{1}{\Delta^3}\int_0^{\Delta}y\,\mathrm{d}y\,A_L(y+\zeta)$$
$$\times\exp(-\mathrm{i}\eta_0 y/\Delta)\int_{\zeta+y}^{\zeta+\Delta}\mathrm{d}z'\,n_0(z')\,.\qquad(25)$$

5. Stationary solution

The stationary solution can be found if we require that at the next pass the laser field envelope be proportional to the initial envelope, i.e.

$$\alpha A_L(s)=(1-\gamma_T)A_L(s,T_e)+(1-\gamma_T)\mathrm{i}\frac{BV}{2\pi}F(s),\qquad(26)$$

where γ_T is the transmission coefficient of the cavity mirrors. Putting $\delta T=T_c-T_e$ and using the periodicity condition of the cavity we obtain

$$A_L(s,T_e)=A_L(s-cT_e)$$
$$=A_L(s-cT_c+c\delta T)=A_L(s+c\delta T)\,,\qquad(27)$$

and if $\delta T\ll T_c$

$$\frac{\alpha+\gamma_T-1}{1-\gamma_T}A_L(s)-c\delta T\frac{\mathrm{d}}{\mathrm{d}s}A_L(s)$$
$$=\mathrm{i}\frac{BV}{2\pi}\frac{c+v_0}{2c}\frac{1}{\Delta^3}\int_0^{\Delta}y\,\mathrm{d}y\,A_L(s+y)$$
$$\times\exp(-\mathrm{i}\eta_0 y/\Delta)\int_{s+y}^{s+\Delta}\mathrm{d}z'\,n_0(z')\,.\qquad(28)$$

This is the equation deduced and numerically integrated in ref. [6].

As a conclusion we have first displayed a space-time picture and shown the qualitative evolution of FEL pulse in a way which allows to choose the correct boundary conditions. Then we have given an analytical solution of a linearized system of differential equations, rederiving results already given in ref. [6] in the single particle treatment. Multiparticle treatments [4] have so far been carried out in terms of numerical solutions.

We notice the role of the phase factor in eq. (25) which gives the dependence of the gain function $F(\zeta)$ on the detuning parameter η_0. In the limit of continuous electron and laser beam $F(\zeta)$ reduces to the single mode gain formula (16).

We want also to stress that the laser envelope in eq. (28) at a point s depends on the same envelope at a point $s+y$, which is the meaning of FEL lethargy.

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