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STOCHASTIC TIME APPROACH TO THE DECAY OF UNSTABLE STATES: FAILURE OF THE ASYMPTOTIC APPROXIMATION ☆

F.T. ARECCHI
Università di Firenze and Istituto Nazionale di Ottica, Florence, Italy

A. POLITI and L. ULIVI
Istituto Nazionale di Ottica, Florence, Italy

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The stochastic time approach allows a closed formulation of the decay of an unstable state. Furthermore it yields a simple analytic formulation in regions where the dynamical evolution can be considered as deterministic. We show here, both theoretically and experimentally, that this approximation fails near the unstable starting point and the stable end point.

Since 1967 experimental evidence of anomalous fluctuations in the decay of unstable states has been given by photon statistics experiments on a transient single-mode laser [1]. More complex is the case of transient decays in extended systems, such as spinodal decomposition [2,3]. However, a mode expansion and the isolation of the most unstable one, allows one to treat the continuous case as a single degree of freedom in the neighbourhood of the lowest instability threshold, as done theoretically e.g. by Haken [4] and experimentally in low angle X-ray scattering [2].

It has been emphasized [5,6] that the decay of unstable states displays the unique feature of having relative fluctuations which do not scale with the reciprocal of the system size, hence showing strong effects even in macorscopic systems.

A stochastic time approach has been introduced independently by two groups [7,8] as an alternative method to the more standard transient amplitude measurements at fixed times. The method consists in classifying the spread $Q(t;b)$ in crossing times t for a given threshold value b rather than classifying the spread $P(x;\bar{t})$ in amplitude x for a given time $\bar{t}$ from the onset time (time of preparation of the unstable system). This is an extended use of the first passage time familiar in chemical processes [9] and thus far essentially limited to the analysis of the first moment of the time distribution (so called reaction times or escape times).

The two approaches that from now on we shall denote by the associated statistical distributions, namely $P(x;\bar{t})$ and $Q(t;b)$, are complementary. They resemble the two approaches to a hydrodynamic flow, namely Lagrange's, consisting in describing single trajectories as if the observer were on top of each particle and then tracing the spread $P(x;\bar{t})$ in single positions at a given time $\bar{t}$, or Euler's, consisting in giving the spread $Q(t;b)$ in arrival times as seen by a single observer fixed at a given position b.

The time spread measurements however offer a more direct information than the amplitude spread, in so far as the associated variance $\Delta T \equiv \langle t^2\rangle - \langle t\rangle^2$ plotted versus the threshold position, is flat whenever the stochasticity is due only to a spread in the initial condition and not to noise along the path [7]. Such a feature is missing if one describes the fluctuations in the $P(x,\bar{t})$ picture as the variance in the stochastic amplitude at fixed times [1,5,6]. In this latter case there is no criterion to discriminate between the effects of initial spread and noise along the path.

A flat time variance (deterministic trajectories starting from random initial conditions) is a good ap-

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proximation for oscillators driven far away from the critical point where gain compensates losses, and in general, for the fall of a point mass in a preassigned potential well when the distance between the unstable maximum and the stable minimum is much bigger than the local jitter due to noise. In other words, a flat time variance is an asymptotic approximation equivalent to considering a deterministic dynamics, perturbed by noise only in the initial position but not along the trajectory. Within this asymptotic approximation, a set of analytic results have been given for the whole distribution $Q(t;b)$ [8].

On the contrary, we have focused the attention on the regions around the equilibrium points where the asymptotic approximation fails. We show this failure in two ways. First we take the laser transient and compute the plots of the variance versus the threshold position, for several values of the so called pump parameter. To be more specific, we modellize the single mode laser as a cubic two-dimensional oscillator of amplitude z driven by white noise, that is

$$\dot{z} = \gamma z - \beta |z|^2 z + \xi \,, \qquad (1)$$

where ξ is a gaussian noise source with zero average and correlation function given by

$$\langle \xi^*(0)\xi(t)\rangle = D\delta(t) \,. \qquad (2)$$

The pump parameter [10] is

$$A = \gamma/(\beta D)^{1/2} \,. \qquad (3)$$

When $A < 0$ ($\gamma < 0$) losses prevail on gain, the nonlinearity plays no role and the oscillator undergoes a standard linear brownian motion with a stationary distribution $p(z)$ being gaussian with zero average. When $A > 0$, gain prevails on losses and for large A the nonlinearity may render noise negligible so that we have practically a deterministic evolution towards the amplitude $|z| = (\gamma/\beta)^{1/2}$.

A transient study is performed by preparing the oscillator with $A < 0$, and then suddenly putting a step increase to $A > 0$. At $t = 0$ the oscillator amplitude is $z_0 \equiv \langle z(0)\rangle = 0$. For large times it reaches a final value

$$z_1 \equiv \langle |z(\infty)|\rangle = (\gamma/\beta)^{1/2} \,. \qquad (4)$$

The plot in fig. 1 is computed by using the moment equations of ref. [7] for the stochastic time corresponding to different threshold settings between z_0 and z. Let us look at the flat region in curves with high A. That means that ΔT does not depend on the position where measurements are performed, that is, all trajectories are shifted versions of one another. The initial and final regions strongly diverge from such approximation, that is, not only the start but also the arrival is badly described by the asymptotic dynamics. This last point was overlooked in ref. [8].

In order to show this experimentally, we have

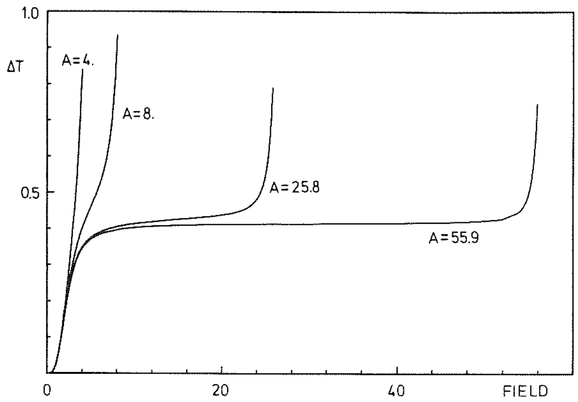

Fig. 1. Model of a laser transient. Plot of variance in stochastic time crossing versus the threshold. The curves correspond to different pump parameter, $A = 4; A = 8; A = 25.8; A = 55.9$. Vertical scale: $\Delta T\gamma^2$ where γ was given in eq. (1). Horizontal scale: threshold amplitude normalized to $(D/\gamma)^{1/2}$, see eq. (1).

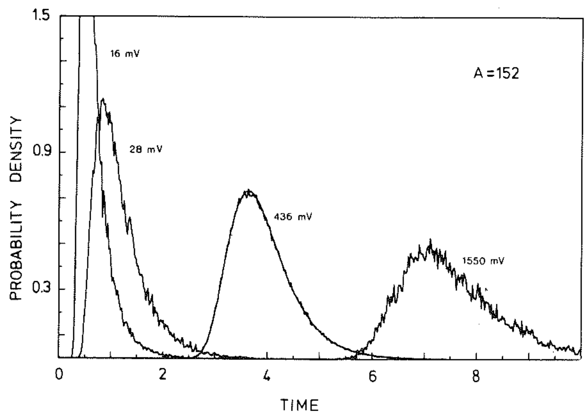

Fig. 2. Experimental time distributions $Q(t;b)$ versus time t normalized to $1/\gamma$. The oscillator equation is as eq. (1) with $\gamma = 807\ s^{-1}$, $\beta = 337\ s^{-1}/V^2$, D is such that $A = 152$. The four curves correspond to the threshold values respectively of $b = 0.016; 0.028; 0.436; 1.550$ V. The wavy lines are experimental records, the smooth line which fits the third curve is a theoretical one computed from eq. (5).

built an electronic oscillator driven by noise and have performed stochastic time measurements giving a gain step from $A < 0$ to $A = 152$. The measuring technique is described in ref. [7]. The plots of fig. 2 are a set of stochastic time probability distributions $Q(t;b)$ measured at different threshold values b. In the asymptotic approximation of ref. [8] we would have expected shifted versions of the same curve

$$Q(t;b) = 2\exp[-2(t-t_0) - e^{-2(t-t_0)}] , \quad (5)$$

where $t_0 = t_0(b)$ is a rigid translation depending on the threshold b. Fig. 2 shows that only in the region which was flat in fig. 1 we can fit our experimental data with eq. (5) as shown by the close agreement between experiment and theory for curve taken at threshold $b = 0.436$ V.

In conclusion, we have shown, both on a laser model and in an experimental case, the limits of validity of the asymptotic approximation of ref. [8]. The initial and final deviations of ΔT from flatness require full use of the moment relations given in ref. [7].

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