*NOISE IN PHYSICAL SYSTEMS AND 1/f NOISE*
*M. Savelli, G. Lecoy and J-P. Nougier (eds.)*



# 1/f SPECTRA IN NONLINEAR SYSTEMS WITH MANY ATTRACTORS

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Power spectra of multistable systems show a low frequency component corresponding to jumps among independent attractors. This situation is modeled with a one-dimensional cubic map disturbed by noise.

Recent experiments [1,2] on nonlinear driven systems yield power spectra with a low frequency divergence $f^{-\alpha}$, $\alpha$ being around 1 whenever the following conditions are fulfilled: i) the system is multistable, that is, it has 2 or more attractors; ii) the attractors are near to be destabilized or they have just become unstable; and iii) the system is "open" to external fluctuations, i.e., the presence of white noise is essential to yield jumps between different basins of attraction. For the nonlinear electronic oscillator of Ref.1, we report in fig.1 the

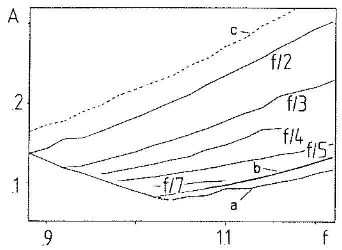


Figure 1 : Experimental parameter space for the system of Ref.1. Under line a the system has stable limit cycles. Above line b there are chaotic attractors covering two valleys, mixed with "islands" of periodic motion as indicated by f/5, f/4 and so on. Line c sets an upper limit due to electronics. 1/f noise is present within the narrow under the thick part of line b.

positions of subharmonic bifurcations, chaotic regions and 1/f region versus amplitude A and frequency f of the driving term.

Usually our power spectra extend over less than two decades, at frequencies much less than the driving frequency. For the $CO_2$ laser [2], we have an $f^{-0.6}$ spectrum between 1 and 100 Hz, while the driving frequency is around 60 KHz.

To understand the above features we have investigated numerically the dynamical equations of the systems of Ref.1 and 2. An integration of the Duffing equation (Ref.1, Eq.(1)) for a suitable choise of parameters allowing for the simultaneous coexistence of four attractors, two period-4 and two period-7, leads to a spectrum (fig.2) showing a power law region extended over about 2 decades with a slope around 1.3. However, numerical integration of a differential system is a lengthy procedure requiring a few hundred steps per period, hence we have preferred to describe the dynamics by a recursive map as done in some approaches to chaotic phenomena [3].

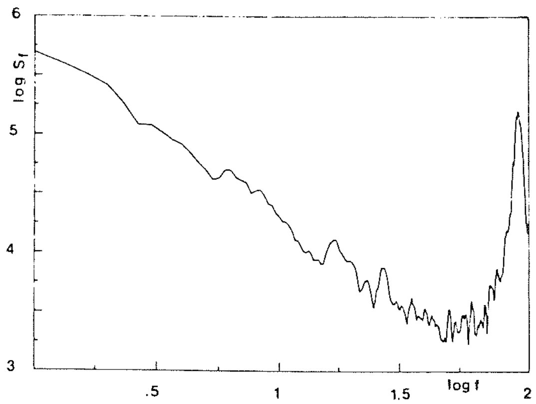


Figure 2 : Power spectrum for the driven Duffing oscillator with k = 0.154, A = 0.114, $\omega$ = 1.22 and spectral density of added noise $6 \times 10^{-6}$.

The above listed three conditions show that we are in presence of a phenomenon which occurs beyond the usual approach to chaos by either one of the current scenarios [4]. A simple jump between two attractors, as introduced in recent models [5], is not sufficient to explain the phenomenology. Indeed experiments [1,2] show that, when leaving an attractor, the representative point in phase space has a long erratic motion before landing onto another attractor. This "transient" regime is made of motions among repulsive orbits.

A dynamics in terms of a recursive map must allow for at least two independent attractors. The simplest one-dimensional map with two attractors must have two extrema [6]. Hence we study a cubic map in the interval (-1, 1)

$$x_{n+1} = (a-1)x_n - ax_n^3. \qquad (1)$$

To account for item iii) the map will be disturbed by white noise with r.m.s. between $10^{-7}$ and $10^{-5}$. Up to a value $a = \bar{a} = 3\sqrt{3}/2 + 1 = 3.598\ 076 \ldots$, the motion is confined either on the interval (-1,0) or (0,1) with qualitative features alike the well known logistic map. For $a = \bar{a}$, we may still have two independent attractors, whose domains, however, are interlaced in complicated ways over the interval (-1,1). For a = 4, even this new structure becomes unstable and there are no longer attractors.

The simplest stable pair of attractors A,B above $\bar{a}$ is a pair of period-3 attractors which are superstable for $a_s = 3.981\ 797\ 394 \ldots$ . These period-3 attractors disappear for $a = \tilde{a} = 3.982\ 000\ 642 \ldots$ . For $a_s < a < \tilde{a}$ (fig. 3) the presence of a small amount of noise makes it easy to leave one attractor and jump toward the other one. Before landing onto the other attractor, the representative point wanders on the available space through a long transient, because of the complex structure of the two basins of attraction. The corresponding low frequency power spectra, for different noise levels, are given in fig. 4.

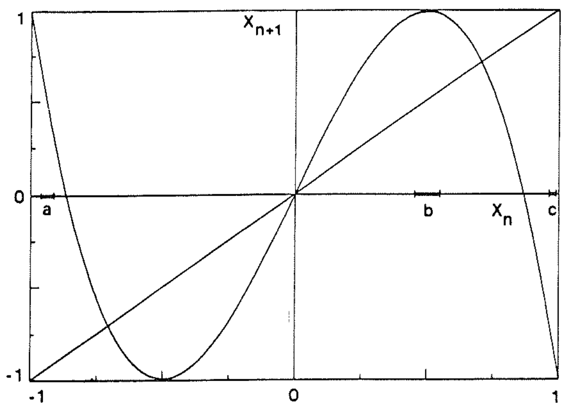


Figure 3: Cubic map for $\tilde{a}$. a,b, and c show (not in scale) the intervals covered by one of the two period-3 attractors.

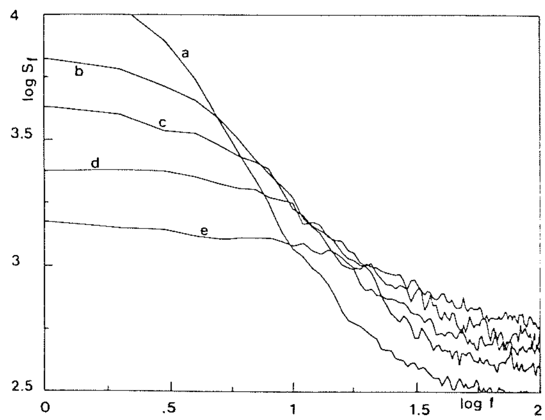


Figure 4 : Power spectra for $a = \tilde{a}$, and for increasing noise levels $\sigma$, that is, a) $5 \times 10^{-7}$, b) $10^{-6}$, c) $2 \times 10^{-6}$, d) $4 \times 10^{-6}$, e) $10^{-5}$. They can be fitted by $f^{-\alpha}$, with $\alpha$ decreasing from 1.5 for a) to 0.5 for e).

These spectra show a power law region extending over about one decade with a slope between 0 and 2. They appear qualitatively in agreement with the experimental spectra of Refs.1 and 2.

Fig. 5 shows the experimental dependence of the Lyapunov exponent on the r.m.s. $\sigma$ of the applied noise for different values of the control parameter a. The relevant $\lambda$ range is confined between the value $\lambda_0 = (\ln 2)/3$ where the two period-3 attractors disappear and $\lambda_T = \ln 3$ where the whole map becomes unstable. The added noise allows to escape from an attracting region whenever the phase point is away from the boundary by less than about $3\sigma$. For any a value, the corresponding $\sigma = \sigma_0$ (a) for which the $\lambda$ curve crosses $\lambda = \lambda_0$ is approximately one third the minimum distance between the border of the region covered by the attractor and the frontier of the immediate attraction domain [7]. Experimentally the dependence is fitted by

$$\sigma_0(a) = \sigma_M \left[1 - \left(\frac{a - a^0}{\tilde{a} - a^0}\right)^{\gamma}\right], \qquad (2)$$

where $\sigma_M = 1.45 \times 10^{-5}$, $a^0 = 3.98184$, $\gamma = 1.40 \pm 0.1$, $a^0 < a < \tilde{a}$. In other words, the jumps between attractors appear as a noise-induced destabilization. $\sigma_T$ is the noise value for which the whole map becomes unstable, hence all the $\lambda$

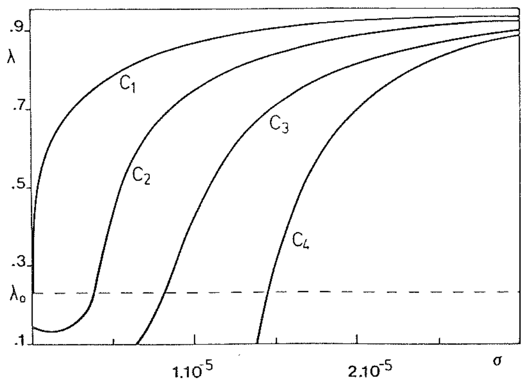


Figure 5 : Lyapunov exponents $\lambda$ vs noise $\sigma$ for different a values. $C_1$:a=$\tilde{a}$, $C_2$ :a=3.98197, $C_3$:a=3.98193, $C_4$:a=$a^0$

curves cluster at $\lambda_T$ for $\sigma = \sigma_T$ with an empirical scaling law $\bar{\lambda} = F(a, \sigma)$ where

$$\bar{\lambda} = (\lambda - \lambda_0)/(\lambda_T - \lambda_0). \qquad (3)$$

During the motion, when the phase point is on an attractor, the Lyapunov exponent is practically equal to $\lambda_0$ , and when it is in the transient region, we can approximately take the asymptotic value $\lambda_T$ . We can then split the sum defining $\lambda$ into separate contributions. Calling $N_A$, $N_T$ the number of steps on the attractor and on the transient respectively, we get

$$\lambda = \lim_{N \to \infty} \left[2 \sum_{i=1}^{N_A} \ln|f'(x_i)| + \sum_{i=1}^{N_T} \ln|f'(x_i)|\right]/N = \qquad (4)$$

$$= 2k\lambda_0 + k'\lambda_T ,$$

where $k = \lim_{N \to \infty} N_A/N$ and $k' = \lim_{N \to \infty} N_T/N$, with $2k + k' = 1$, are the fractional times spent on the attractor and on the transient, respectively. Notice from eqs.(3) and (4) that the fractional time k' spent on the transient coincides with the reduced Lyapunov exponent $\bar{\lambda}$ given in the scaling law. Inspection of the date of fig.4 and fig.5 shows that the slopes $\alpha$ of the $f^{-\alpha}$ regions scale linearly with $\bar{\lambda}$ as $\alpha = 2(1 - \bar{\lambda})$, from $\alpha = 2$ (single Lorentzian), when $\bar{\lambda} = 0$ (no transient), to $\alpha \to 0$ (flat spectrum) for $\bar{\lambda} = 1$ (all transient).

A model explanation of the above spectra was given[ 8 ] in terms of jump processes among three regions of phase space (the two attractors and the intermediate transient), by taking the jump probabilities decorrelated from the internal motions within the three regions. This leads to a power spectrum made of three Lorentzians which fit well the spectra shown in fig.4.

In the particular case of equal transition probabilities and nearest neighbors jumps between attractors, we obtain a low frequency spectrum which approaches 1/f. Our hopping mechanism does not explain in general 1/f noise in linear equilibrium situations. It opens, however, a new area of investigation, namely that of noise induced interactions among many attractor domains.

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[1] F.T. Arecchi and F. Lisi, Phys. Rev. Lett. 49 (1982) 94. The slopes in the spectra of this paper must be multiplied by 2, for a miscalibration of the spectrum analyzer.

[2] F.T. Arecchi, R. Meucci, G. Puccioni and J. Tredicce, Phys. Rev. Lett. 49 (1982) 1217.

[3] E. Ott, Rev. Mod. Phys. 53 (1981) 643.

[4] J.P. Eckmann, Rev. Mod. Phys., 53 (1981) 743.

[5] E. Ben Jacob, I. Goldhirsch, Y. Imry and S. Fishman, Phys. Rev. Lett. 49 (1982) 1599.

[6] I. Gumowski and C. Mira: dynamique chaotique (Cepadues Editions, Toulouse, 1980).

[7] The immediate basin of attraction is defined (Ref. 6) as the neighborhood of an attractor within which the distance from the latter shrinks monotonically.

[8] F.T. Arecchi, R. Badii and A. Politi, to be published.